

Riemann-Roch and index formulae in twisted K -theory

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ABSTRACT. In this paper, we establish the Riemann-Roch theorem in twisted K -theory extending our earlier results. We also give a careful summary of twisted geometric cycles explaining in detail some subtle points in the theory. As an application, we prove a twisted index formula and show that D-brane charges in Type I and Type II string theory are classified by twisted KO-theory and twisted K -theory respectively in the presence of B -fields as proposed by Witten.

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1991 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.

Key words and phrases. Twisted K -theory, twisted K -homology, twisted Riemann-Roch, twisted index theorem, D-brane charges.

1. Introduction

1.1. String geometry. We begin by giving a short discussion of the physical background. Readers uninterested in this motivation may move to the next subsection. In string theory D-branes were proposed as a mechanism for providing boundary conditions for the dynamics of open strings moving in space-time. Initially they were thought of as submanifolds. As D-branes themselves can evolve over time one needs to study equivalence relations on the set of D-branes. An invariant of the equivalence class is the topological charge of the D-brane which should be thought of as an analogue of the Dirac monopole charge as these D-brane charges are associated with gauge fields (connections) on vector bundles over the D-brane. These vector bundles are known as Chan-Paton bundles.

In [MM] Minasian and Moore made the proposal that D-brane charges should take values in K -groups and not in the cohomology of the space-time or the D-brane. However, they proposed a cohomological formula for these charges which might be thought of as a kind of index theorem in the sense that, in general, index theory associates to a K -theory class a number which is given by an integral of a closed differential form. In string theory there is an additional field on space-time known as the H -flux which may be thought of as a global closed three form. Locally it is given by a family of ‘two-form potentials’ known as the B -field. Mathematically we think of these B -fields as defining a degree three integral Čech class on the space-time, called a ‘twist’. Witten [Wit], extending [MM], gave a physical argument for the idea that D-brane charges should be elements of K -groups and, in addition, proposed that the D-brane charges in the presence of a twist should take values in twisted K -theory (at least in the case where the twist is torsion). The mathematical ideas he relied on were due to Donovan and Karoubi [DK]. Subsequently Bouwknegt and Mathai [BouMat] extended Witten’s proposal to the non-torsion case using ideas from [Ros]. A geometric model (that is, a ‘string geometry’ picture) for some of these string theory constructions and for twisted K -theory was proposed in [BCMMS] using the notion of bundle gerbes and bundle gerbe modules. Various refinements of twisted K -theory that are suggested by these applications are also described in the article of Atiyah and Segal [AS1] and we will need to use their results here.

1.2. Mathematical results. From a mathematical perspective some immediate questions arise from the physical input summarised above. When there is no twist it is well known that K -theory provides the main topological tool for the index theory of elliptic operators. One version of the Atiyah-Singer index theorem due to Baum-Higson-Schick [BHS] establishes a relationship between the analytic viewpoint provided by elliptic differential operators and the geometric viewpoint provided by the notion of geometric cycle introduced in the fundamental paper of Baum and Douglas [BD2]. The viewpoint that geometric cycles in the sense of [BD2] are a model for D-branes in the untwisted case is expounded in [RS, RSV, Sz]. Note that in this viewpoint D-branes are no longer submanifolds but the images of manifolds under a smooth map.

It is thus tempting to conjecture that there is an analogous picture of D-branes as a type of geometric cycle in the twisted case as well. More precisely we ask the question of whether there is a way to formulate the notion of ‘twisted geometric cycle’ (cf [BD1] and [BD2]) and to prove an index theorem in the spirit of [BHS] for twisted K -theory. This precise question was answered in the positive in [Wa]. It is important to emphasise that string geometry ideas from [FreWit] played a key role in finding the correct way to generalise [BD1].

Our purpose here in the present paper is threefold. First, we explain the results in [Wa] (see Section 5) in a fashion that is more aligned to the string geometry viewpoint. Second, we prove an analogue of the Atiyah-Hirzebruch Riemann-Roch formula in twisted K -theory by extending the results and approach of [CMW]. An interesting by-product of our approach in Section 5 is a discussion of the Thom class in twisted K theory. Third, in Section 6 we prove an index formula using our twisted Riemann-Roch theorem. It will be clear from our approach to this twisted index theory that our twisted geometric cycles provide a geometric model for D-branes and we give details in Section 7.

Our main new results are stated as two theorems, Theorem 5.3 (twisted Riemann-Roch) and Theorem 6.1 (the index pairing). We remark that the Minasian-Moore formula [MM] arises from the fact that the index pairing they discuss may be regarded as a quadratic form on K -theory. In the twisted index formula that we establish, the pairing is asymmetric and may be thought of as a bilinear form, from which there is no obvious way to extract a twisted analogue of the Minasian-Moore formula. Nevertheless we interpret our results in terms of the physics language in Section 7 explaining the link to Witten's original ideas on D-brane charges.

2. Twisted K -theory: preliminary review

2.1. Twisted K -theory: topological and analytic definitions. We begin by reviewing the notion of a ‘twisting’. Let \mathcal{H} be an infinite dimensional, complex and separable Hilbert space. We shall consider locally trivial principal $PU(\mathcal{H})$ -bundles over a paracompact Hausdorff topological space X , the structure group $PU(\mathcal{H})$ is equipped with the norm topology. The projective unitary group $PU(\mathcal{H})$ with the topology induced by the norm topology on $U(\mathcal{H})$ (Cf. [Kui]) has the homotopy type of an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. The classifying space of $PU(\mathcal{H})$, denoted $BPU(\mathcal{H})$, is a $K(\mathbb{Z}, 3)$. The set of isomorphism classes of principal $PU(\mathcal{H})$ -bundles over X is given by (Proposition 2.1 in [AS1]) homotopy classes of maps from X to any $K(\mathbb{Z}, 3)$ and there is a canonical identification

$$[X, BPU(\mathcal{H})] \cong H^3(X, \mathbb{Z}).$$

A twisting of complex K -theory on X is given by a continuous map $\alpha : X \rightarrow K(\mathbb{Z}, 3)$. For such a twisting, we can associate a canonical principal $PU(\mathcal{H})$ -bundle \mathcal{P}_α through the usual pull-back construction from the universal $PU(\mathcal{H})$ bundle denoted by $EK(\mathbb{Z}, 2)$, as summarised by the diagram:

$$(2.1) \quad \begin{array}{ccc} \mathcal{P}_\alpha & \longrightarrow & EK(\mathbb{Z}, 2) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3). \end{array}$$

We will use $PU(\mathcal{H})$ as a group model for a $K(\mathbb{Z}, 2)$. We write $\mathbf{Fred}(\mathcal{H})$ for the connected component of the identity of the space of Fredholm operators on \mathcal{H} equipped with the norm topology. There is a base-point preserving action of $PU(\mathcal{H})$ given by the conjugation action of $U(\mathcal{H})$ on $\mathbf{Fred}(\mathcal{H})$:

$$(2.2) \quad PU(\mathcal{H}) \times \mathbf{Fred}(\mathcal{H}) \longrightarrow \mathbf{Fred}(\mathcal{H}).$$

The action (2.2) defines an associated bundle over X which we denote by

$$\mathcal{P}_\alpha(\mathbf{Fred}) = \mathcal{P}_\alpha \times_{PU(\mathcal{H})} \mathbf{Fred}(\mathcal{H})$$

We write $\{\Omega_X^n \mathcal{P}_\alpha(\mathbf{Fred}) = \mathcal{P}_\alpha \times_{PU(\mathcal{H})} \Omega^n \mathbf{Fred}\}$ for the fiber-wise iterated loop spaces.

$$K^{-n}(X, \alpha) := \pi_0(C_c(X, \Omega_X^n \mathcal{P}_\alpha(\mathbf{Fred}))),$$

Due to Bott periodicity, we only have two different twisted K-groups $K^0(X, \alpha)$ and $K^1(X, \alpha)$. Given a closed subspace A of X , then (X, A) is a pair of topological spaces, and we define relative twisted K-groups to be

$$K^{ev/odd}(X, A; \alpha) := K^{ev/odd}(X - A, \alpha).$$

$$\begin{array}{ccc}
& \alpha_0 & \\
X & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \eta & K(\mathbb{Z}, 3) \\
& \alpha_1 &
\end{array}$$
$$(2.3) \quad \eta_* : K^{ev/odd}(X, \alpha_0) \xrightarrow{\cong} K^{ev/odd}(X, \alpha_1),$$
$$\mathbf{Vect}_1(X) \cong [X, K(\mathbb{Z}, 2)] \cong H^2(X, \mathbb{Z})$$

Let \mathcal{K} be the C^* -algebra of compact operators on \mathcal{H} . The isomorphism $PU(\mathcal{H}) \cong Aut(\mathcal{K})$ via the conjugation action of the unitary group $U(\mathcal{H})$ provides an action of a $K(\mathbb{Z}, 2)$ on the C^* -algebra \mathcal{K} . Hence, any $K(\mathbb{Z}, 2)$ -principal bundle \mathcal{P}_α defines a locally trivial bundle of compact operators, denoted by $\mathcal{P}_\alpha(\mathcal{K}) = \mathcal{P}_\alpha \times_{PU(\mathcal{H})} \mathcal{K}$.

Let $C_0(X, \mathcal{P}_\alpha(\mathcal{K}))$ be the C^* -algebra of sections of $\mathcal{P}_\alpha(\mathcal{K})$ vanishing at infinity. Then $C_0(X, \mathcal{P}_\alpha(\mathcal{K}))$ is the (unique up to isomorphism) stable separable complex continuous-trace C^* -algebra over X with Dixmier-Douady class $[\alpha] \in H^3(X, \mathbb{Z})$ (here we identify the Čech cohomology of X with its singular cohomology, cf [Ros] and [AS1]).

THEOREM 2.2. ([AS1] and [Ros]) *The topological twisted K-groups $K^{ev/odd}(X, \alpha)$ are canonically isomorphic to analytic K-theory of the C^* -algebra $C_0(X, \mathcal{P}_\alpha(\mathcal{K}))$*

$$K^{ev/odd}(X, \alpha) \cong K_{ev/odd}(C_0(X, \mathcal{P}_\alpha(\mathcal{K})))$$

where the latter group is the algebraic K -theory of $C_0(X, \mathcal{P}_\alpha(\mathcal{K}))$, defined to be

$$\varinjlim_{k \rightarrow \infty} \pi_1(GL_k(C_0(X, \mathcal{P}_\alpha(\mathcal{K}))))).$$

Note that the algebraic K -theory of $C_0(X, \mathcal{P}_\alpha(\mathcal{K}))$ is isomorphic to Kasparovs KK -theory ([Kas1] and [Kas2])

$$KK^{ev/odd}(\mathbb{C}, C_0(X, \mathcal{P}_\alpha(\mathcal{K}))).$$

It is important to recognise that these groups are only defined up to isomorphism by the Dixmier-Douady class $[\alpha] \in H^3(X, \mathbb{Z})$. To distinguish these two equivalent definitions of twisted K -theory if needed, we will write

$$K_{\text{top}}^{\text{ev/odd}}(X, \alpha) \quad \text{and} \quad K_{\text{an}}^{\text{ev/odd}}(X, \alpha)$$

for the topological and analytic twisted K -theories of (X, α) respectively. Twisted K -theory is a 2-periodic *generalized cohomology theory*: a contravariant functor on the category consisting of pairs (X, α) , with the twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$, to the category of \mathbb{Z}_2 -graded abelian groups. Note that a morphism between two pairs (X, α) and (Y, β) is a continuous map $f : X \rightarrow Y$ such that $\beta \circ f = \alpha$.

2.2. Twisted K -theory for torsion twistings. There are some subtle issues in twisted K -theory and to handle these we have chosen to use the language of bundle gerbes, connections and curvings as explained in [Mur]. We explain first the so-called ‘lifting bundle gerbe’ \mathcal{G}_α [Mur] associated to the principal $PU(\mathcal{H})$ -bundle $\pi : \mathcal{P}_\alpha \rightarrow X$ and the central extension

$$(2.4) \quad 1 \rightarrow U(1) \longrightarrow U(\mathcal{H}) \longrightarrow PU(\mathcal{H}) \rightarrow 1.$$

This is constructed by starting with $\pi : \mathcal{P}_\alpha \rightarrow X$, forming the fibre product $\mathcal{P}_\alpha^{[2]}$ which is a groupoid

$$\mathcal{P}_\alpha^{[2]} = \mathcal{P}_\alpha \times_X \mathcal{P}_\alpha \xrightleftharpoons[\pi_2]{\pi_1} \mathcal{P}_\alpha$$

with source and range maps $\pi_1 : (y_1, y_2) \mapsto y_1$ and $\pi_2 : (y_1, y_2) \mapsto y_2$. There is an obvious map from each fiber of $\mathcal{P}_\alpha^{[2]}$ to $PU(\mathcal{H})$ and so we can define the fiber of \mathcal{G}_α over a point in $\mathcal{P}_\alpha^{[2]}$ by pulling back the fibration (2.4) using this map. This endows \mathcal{G}_α with a groupoid structure (from the multiplication in $U(\mathcal{H})$) and in fact it is a $U(1)$ -groupoid extension of $\mathcal{P}_\alpha^{[2]}$.

A torsion twisting α is a map $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ representing a torsion class in $H^3(X, \mathbb{Z})$. Every torsion twisting arises from a principal $PU(n)$ -bundle $\mathcal{P}_\alpha(n)$ with its classifying map

$$X \rightarrow BPU(n),$$

or a principal $PU(\mathcal{H})$ -bundle with a reduction to $PU(n) \subset PU(\mathcal{H})$. For a torsion twisting $\alpha : X \rightarrow BPU(n) \rightarrow BPU(\mathcal{H})$, the corresponding lifting bundle gerbe \mathcal{G}_α

$$(2.5) \quad \begin{array}{ccc} \mathcal{G}_\alpha & & \\ \downarrow & & \\ \mathcal{P}_\alpha(n)^{[2]} & \xrightleftharpoons[\pi_2]{\pi_1} & \mathcal{P}_\alpha(n) \\ & & \downarrow \pi \\ & & M \end{array}$$

is defined by $\mathcal{P}_\alpha(n)^{[2]} \cong \mathcal{P}_\alpha(n) \rtimes PU(n) \rightrightarrows \mathcal{P}_\alpha(n)$ (as a groupoid) and the central extension

$$1 \rightarrow U(1) \longrightarrow U(n) \longrightarrow PU(n) \rightarrow 1.$$

There is an Azumaya bundle associated to $\mathcal{P}_\alpha(n)$ arising naturally from the $PU(n)$ action on the $n \times n$ matrices. We denote this associated Azumaya bundle by \mathcal{A}_α . An

\mathcal{A}_α -module is a complex vector bundle \mathcal{E} over M with a fiberwise \mathcal{A}_α action

$$\mathcal{A}_\sigma \times_M \mathcal{E} \longrightarrow \mathcal{E}.$$

The C^* -algebra of continuous sections of \mathcal{A}_α , vanishing at infinity if X is non-compact, is Morita equivalent to a continuous trace C^* -algebra $C_0(X, \mathcal{P}_\alpha(\mathcal{K}))$. Hence there is an isomorphism between $K^0(X, \alpha)$ and the K -theory of the bundle modules of \mathcal{A}_α .

There is an equivalent definition of twisted K -theory using bundle gerbe modules (Cf. [BCMMS] and [CW1]). A bundle gerbe module E of \mathcal{G}_α is a complex vector bundle E over $\mathcal{P}_\alpha(n)$ with a groupoid action of \mathcal{G}_α , i.e., an isomorphism

$$\phi : \mathcal{G}_\alpha \times_{(\pi_2, p)} E \longrightarrow E$$

where $\mathcal{G}_\alpha \times_{(\pi_2, \pi)} E$ is the fiber product of the source $\pi_2 : \mathcal{G}_\alpha \rightarrow \mathcal{P}_\alpha(n)$ and $p : E \rightarrow \mathcal{P}_\alpha(n)$ such that

- (1) $p \circ \phi(g, v) = \pi_1(g)$ for $(g, v) \in \mathcal{G}_\alpha \times_{(\pi_2, p)} E$, and π_1 is the target map of \mathcal{G}_α .
- (2) ϕ is compatible with the bundle gerbe multiplication $m : \mathcal{G}_\alpha \times_{(\pi_2, \pi_1)} \mathcal{G}_\alpha \rightarrow \mathcal{G}_\alpha$, which means

$$\phi \circ (id \times \phi) = \phi \circ (m \times id).$$

Note that the natural representation of $U(n)$ on \mathbb{C}^n induces a \mathcal{G}_α bundle gerbe module

$$S_n = \mathcal{P}_\alpha(n) \times \mathbb{C}^n.$$

Here we use the fact that $\mathcal{G}_\alpha = \mathcal{P}_\alpha(n) \rtimes U(n) \rightrightarrows \mathcal{P}_\alpha(n)$ (as a groupoid). Similarly, the dual representation of $U(n)$ on \mathbb{C}^n induces a $\mathcal{G}_{-\alpha}$ bundle gerbe module $S_n^* = \mathcal{P}_\alpha(n) \times \mathbb{C}^n$. Note that $S_n^* \otimes S_n \cong \pi^* \mathcal{A}_\alpha$ descends to the Azumaya bundle \mathcal{A}_α . Given a \mathcal{G}_α bundle gerbe module E of rank k , then as a $PU(n)$ -equivariant vector bundle, $S_n^* \otimes E$ descends to an \mathcal{A}_α -bundle over M . Conversely, given an \mathcal{A}_α -bundle \mathcal{E} over M , $S_n \otimes_{\pi^* \mathcal{A}_\alpha} \pi^* \mathcal{E}$ defines a \mathcal{G}_α bundle gerbe module. These two constructions are inverse to each other due to the fact that

$$S_n^* \otimes (S_n \otimes_{\pi^* \mathcal{A}_\alpha} \pi^* \mathcal{E}) \cong (S_n^* \otimes S_n) \otimes_{\pi^* \mathcal{A}_\alpha} \pi^* \mathcal{E} \cong \pi^* \mathcal{A}_\alpha \otimes_{\pi^* \mathcal{A}_\alpha} \pi^* \mathcal{E} \cong \pi^* \mathcal{E}.$$

Therefore, there is a natural equivalence between the category of \mathcal{G}_α bundle gerbe modules and the category of \mathcal{A}_α bundle modules, as discussed in [CW1]. In summary, we have the following proposition.

PROPOSITION 2.3. ([BCMMS][CW1]) *For a torsion twisting $\alpha : X \rightarrow BPU(n) \rightarrow BPU(\mathcal{H})$, twisted K -theory $K^0(X, \alpha)$ has another two equivalent descriptions:*

- (1) *the Grothendieck group of the category of \mathcal{G}_α bundle gerbe modules.*
- (2) *the Grothendieck group of the category of \mathcal{A}_σ bundle modules.*

One important example of torsion twistings comes from real oriented vector bundles. Consider an oriented real vector bundle E of even rank over X with a fixed fiberwise inner product. Denote by

$$\nu_E : X \rightarrow \mathbf{BSO}(2k)$$

the classifying map of E . The following twisting

$$o(E) := W_3 \circ \nu_E : X \longrightarrow \mathbf{BSO}(2k) \longrightarrow K(\mathbb{Z}, 3),$$

will be called the orientation twisting associated to E . Here W_3 is the classifying map of the principal $\mathbf{BU}(1)$ -bundle $\mathbf{BSpin}^c(2k) \rightarrow \mathbf{BSO}(2k)$. Note that the orientation twisting $o(E)$ is null-homotopic if and only if E is K-oriented.

PROPOSITION 2.4. *Given an oriented real vector bundle E of even rank over X with an orientation twisting $o(E)$, then there is a canonical isomorphism*

$$K^0(X, o(E)) \cong K^0(X, W_3(E))$$

where $K^0(X, W_3(E))$ is the K -theory of the Clifford modules associated to the bundle $\mathbf{Cliff}(E)$ of Clifford algebras.

PROOF. Denote by $\mathcal{F}r$ the frame bundle of V , the principal $SO(2k)$ -bundle of positively oriented orthonormal frames, i.e.,

$$E = \mathcal{F}r \times_{\rho_{2n}} \mathbb{R}^{2k},$$

where ρ_n is the standard representation of $SO(2k)$ on \mathbb{R}^n . The lifting bundle gerbe associated to the frame bundle and the central extension

$$1 \rightarrow U(1) \longrightarrow Spin^c(2k) \longrightarrow SO(2k) \rightarrow 1$$

is called the $Spin^c$ bundle gerbe $\mathcal{G}_{W_3(E)}$ of E , whose Dixmier-Douady invariant is given by the integral third Stiefel-Whitney class $W_3(E) \in H^3(X, \mathbb{Z})$. The canonical representation of $Spin^c(2k)$ gives a natural inclusion

$$Spin^c(2k) \subset U(2^k)$$

which induces a commutative diagram

$$\begin{array}{ccccc} U(1) & \longrightarrow & Spin^c(2k) & \longrightarrow & SO(2k) \\ \downarrow = & & \downarrow & & \downarrow \\ U(1) & \longrightarrow & U(2^k) & \longrightarrow & PU(2^k) \\ \downarrow = & & \downarrow & & \downarrow \\ U(1) & \longrightarrow & U(\mathcal{H}) & \longrightarrow & PU(\mathcal{H}). \end{array}$$

This provides a reduction of the principal $PU(\mathcal{H})$ -bundle $\mathcal{P}_{o(E)}$. The associated bundle of Azumaya algebras is in fact the bundle of Clifford algebras, whose bundle modules are called Clifford modules ([BGV]). Hence, there exists a canonical isomorphism between $K^0(X, o(E))$ and the K -theory of the Clifford modules associated to the bundle $\mathbf{Cliff}(E)$. \square

2.3. Twisted K -theory: general properties. Twisted K -theory satisfies the following properties whose proofs are rather standard for a 2-periodic generalized cohomology theory ([AS1] [CW1] [Kar] [Wa]). (Note that when we write (X, A) for a pair of spaces we assume $A \subset X$.)

- (I) **(The homotopy axiom)** If two morphisms $f, g : (Y, B) \rightarrow (X, A)$ are homotopic through a map $\eta : (Y \times [0, 1], B \times [0, 1]) \rightarrow (X, A)$, written in terms of the following homotopy commutative diagram

$$\begin{array}{ccc} (Y, B) & \xrightarrow{f} & (X, A) \\ g \downarrow & \eta \nearrow & \downarrow \alpha \\ (X, A) & \xrightarrow{\alpha} & K(\mathbb{Z}, 3), \end{array}$$

then we have the following commutative diagram

$$\begin{array}{ccc}
 & K^{ev/odd}(X, A; \alpha) & \\
 f^* \swarrow & & \searrow g^* \\
 K^{ev/odd}(Y, B; \alpha \circ f) & \xrightarrow{\eta_*} & K^{ev/odd}(Y, B; \alpha \circ g).
 \end{array}$$

Here η_* is the canonical isomorphism induced by the homotopy η .

- (II) **(The exact axiom)** For any pair (X, A) with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$, there exists the following six-term exact sequence

$$\begin{array}{ccccc}
 K^0(X, A; \alpha) & \longrightarrow & K^0(X, \alpha) & \longrightarrow & K^0(A, \alpha|_A) \\
 \uparrow & & & & \downarrow \\
 K^1(A, \alpha|_A) & \longleftarrow & K^1(X, \alpha) & \longleftarrow & K^1(X, A; \alpha)
 \end{array}$$

here $\alpha|_A$ is the composition of the inclusion and α .

- (III) **(The excision axiom)** Let (X, A) be a pair of spaces and let $U \subset A$ be a subspace such that the closure \overline{U} is contained in the interior of A . Then the inclusion $\iota : (X - U, A - U) \rightarrow (X, A)$ induces, for all $\alpha : X \rightarrow K(\mathbb{Z}, 3)$, an isomorphism

$$K^{ev/odd}(X, A; \alpha) \longrightarrow K^{ev/odd}(X - U, A - U; \alpha \circ \iota).$$

- (IV) **(Multiplicative property)** Let $\alpha, \beta : X \rightarrow K(\mathbb{Z}, 3)$ be a pair of twistings on X . Denote by $\alpha + \beta$ the new twisting defined by the following map¹

$$(2.6) \quad \alpha + \beta : X \xrightarrow{(\alpha, \beta)} K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 3) \xrightarrow{m} K(\mathbb{Z}, 3),$$

where m is defined as follows

$$BPU(\mathcal{H}) \times BPU(\mathcal{H}) \cong B(PU(\mathcal{H}) \times PU(\mathcal{H})) \longrightarrow BPU(\mathcal{H}),$$

for a fixed isomorphism $\mathcal{H} \otimes \mathcal{H} \cong \mathcal{H}$. Then there is a canonical multiplication

$$(2.7) \quad K^{ev/odd}(X, \alpha) \times K^{ev/odd}(X, \beta) \longrightarrow K^{ev/odd}(X, \alpha + \beta),$$

which defines a $K^0(X)$ -module structure on twisted K-groups $K^{ev/odd}(X, \alpha)$.

- (V) **(Thom isomorphism)** Let $\pi : E \rightarrow X$ be an oriented real vector bundle of rank k over X , then there is a canonical isomorphism, for any twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$,

$$(2.8) \quad K^{ev/odd}(X, \alpha + o_E) \cong K^{ev/odd}(E, \alpha \circ \pi),$$

with the grading shifted by $k \pmod{2}$.

- (VI) **(The push-forward map)** For any differentiable map $f : X \rightarrow Y$ between two smooth manifolds X and Y , let $\alpha : Y \rightarrow K(\mathbb{Z}, 3)$ be a twisting. Then there is a canonical push-forward homomorphism

$$(2.9) \quad f_!^K : K^{ev/odd}(X, (\alpha \circ f) + o_f) \longrightarrow K^{ev/odd}(Y, \alpha),$$

with the grading shifted by $n \pmod{2}$ for $n = \dim(X) + \dim(Y)$. Here o_f is the orientation twisting corresponding to the bundle $TX \oplus f^*TY$ over X .

¹In terms of bundles of projective Hilbert space, this operation corresponds to the Hilbert space tensor product, see [AS1].

(VII) (**Mayer-Vietoris sequence**) If X is covered by two open subsets U_1 and U_2 with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$, then there is a Mayer-Vietoris exact sequence

$$\begin{array}{ccccccc} K^0(X, \alpha) & \longrightarrow & K^1(U_1 \cap U_2, \alpha_{12}) & \longrightarrow & K^1(U_1, \alpha_1) \oplus K^1(U_2, \alpha_2) \\ \uparrow & & & & \downarrow \\ K^0(U_1, \alpha_1) \oplus K^0(U_2, \alpha_2) & \longleftarrow & K^0(U_1 \cap U_2, \alpha_{12}) & \longleftarrow & K^1(X, \alpha) \end{array}$$

where α_1, α_2 and α_{12} are the restrictions of α to U_1, U_2 and $U_1 \cap U_2$ respectively.

3. Twisted K -homology

Complex K -theory, as a generalized cohomology theory on a CW complex, is developed by Atiyah-Hirzebruch using complex vector bundles. It is representable in the sense that there exists a classifying space $\mathbb{Z} \times BU(\infty)$, where $BU(\infty) = \varinjlim_k BU(k)$, such that

$$K^0(X) = [X, \mathbb{Z} \times BU(\infty)]$$

for any finite CW complex X . The classifying space for complex K -theory is referred to as the $BU(\infty)$ -spectrum with even term $\mathbb{Z} \times BU(\infty)$ and odd term $U(\infty)$. They are also called the ‘complex K -spectra’ in the literature. The advantage of using spectra is that there is a natural definition of a homology theory associated to a classifying space of each generalized cohomology theory. Hence, the topological K -homology of a CW complex X , dual to complex K -theory, is defined by the following stable homotopy groups

$$K_{ev}^{\text{top}}(X) = \varinjlim_{k \rightarrow \infty} \pi_{2k}(BU(\infty) \wedge X^+)$$

and

$$K_{odd}^{\text{top}}(X, \alpha) = \varinjlim_{k \rightarrow \infty} \pi_{2k+1}(BU(\infty) \wedge X^+).$$

Here X^+ is the space X with one point added as a based point, and the wedge product of two based CW complexes (X, x_0) and (Y, y_0) is defined to be

$$X \wedge Y = \frac{X \times Y}{(X \times \{y_0\} \cup \{x_0\} \times Y)}.$$

All the properties of K -homology, as a generalized homology theory, can be obtained in a natural way see for example in [Swi]. There are two other equivalent definitions of K -homology, called analytic K -homology developed by Kasparov, and geometric K -homology by Baum and Douglas. We now give a brief review of these two definitions.

Kasparov’s analytic K -homology $KK^{ev/odd}(C(X), \mathbb{C})$ is generated by unitary equivalence classes of (graded) Fredholm modules over $C(X)$ modulo an operator homotopy relation ([Kas1] and [HigRoe]). For brevity we will use the notation $K_{ev/odd}^{\text{an}}(X)$ for this K -homology. A cycle for $K_0^{\text{an}}(X)$, also called a \mathbb{Z}_2 -graded Fredholm module, consists of a triple $(\phi_0 \oplus \phi_1, \mathcal{H}_0 \oplus \mathcal{H}_1, F)$, where

- $\phi_i : C(X) \rightarrow B(\mathcal{H}_i)$ is a representation of $C(X)$ on a separable Hilbert space \mathcal{H}_i ;
- $F : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ is a bounded operator such that

$$\phi_1(a)F - F\phi_0(a), \quad \phi_0(a)(F^*F - Id) \quad \phi_1(a)(FF^* - Id)$$

are compact operators for all $a \in C(X)$.

A cycle for $K_1^{\text{an}}(X)$, also called a trivially graded or odd Fredholm module, consists of a pair (ϕ, F) , where

- $\phi : C(X) \rightarrow B(\mathcal{H})$ is a representation of $C(X)$ on a separable Hilbert space \mathcal{H} ;
- F is a bounded self-adjoint operator on \mathcal{H} such that

$$\phi(a)F - F\phi(a), \quad \phi(a)(F^2 - Id)$$

are compact operators for all $a \in C(X)$.

In [BD1] and [BD2], Baum and Douglas gave a geometric definition of K -homology using what are now called geometric cycles. The basic cycles for $K_{ev}^{\text{geo}}(X)$ (respectively $K_{odd}^{\text{geo}}(X)$) are triples

$$(M, \iota, E)$$

consisting of even-dimensional (resp. odd-dimensional) closed smooth manifolds M with a given $Spin^c$ structure on the tangent bundle of M together with a continuous map $\iota : M \rightarrow X$ and a complex vector bundle E over M . The equivalence relation on the set of all cycles is generated by the following three steps (see [BD1] for details):

- (i) Bordism.
- (ii) Direct sum and disjoint union.
- (iii) Vector bundle modification.

Addition in $K_{ev/odd}^{\text{geo}}(X)$ is given by the disjoint union operation of geometric cycles.

Baum-Douglas in [BD2] showed that the Atiyah-Singer index theorem is encoded in the following commutative diagram

$$(3.1) \quad \begin{array}{ccc} & K_{ev/odd}^{\text{top}}(X) & \\ \cong \swarrow & & \searrow \cong \\ K_{ev/odd}^{\text{geo}}(X) & \xrightarrow{\mu} & K_{ev/odd}^{\text{an}}(X) \end{array}$$

where μ is the assembly map assigning an abstract Dirac operator

$$\iota_*([\mathcal{D}_M^E]) \in K_{ev/odd}^{\text{an}}(X)$$

to a geometric cycle (M, ι, E) .

For a paracompact Hausdorff space X with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$, all these three versions of twisted K -homology were studied in [Wa]. They are called there the twisted topological, analytic and geometric K -homologies, and denoted respectively by $K_{ev/odd}^{\text{top}}(X, \alpha)$, $K_{ev/odd}^{\text{an}}(X, \alpha)$ and $K_{ev/odd}^{\text{geo}}(X, \alpha)$. Our first task in this Section is to review these three definitions, see [Wa] for greater detail.

3.1. Topological and analytic definitions of twisted K -homology. Let X be a CW complex (or paracompact Hausdorff space) with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$. Let \mathcal{P}_α be the corresponding principal $K(\mathbb{Z}, 2)$ -bundle. Any base-point preserving action of a $K(\mathbb{Z}, 2)$ on a space defines an associated bundle by the standard construction. In particular, as a classifying space of complex line bundles, a $K(\mathbb{Z}, 2)$ acts on the complex K -theory spectrum \mathbb{K} representing the tensor product by complex line bundles, where

$$\mathbb{K}_{ev} = \mathbb{Z} \times BU(\infty), \quad \mathbb{K}_{odd} = U(\infty).$$

Denote by $\mathcal{P}_\alpha(\mathbb{K}) = \mathcal{P}_\alpha \times_{K(\mathbb{Z}, 2)} \mathbb{K}$ the bundle of based K -theory spectra over X . There is a section of $\mathcal{P}_\alpha(\mathbb{K}) = \mathcal{P}_\alpha \times_{K(\mathbb{Z}, 2)} \mathbb{K}$ defined by taking the base points of each fiber. The image of this section can be identified with X and we denote by $\mathcal{P}_\alpha(\mathbb{K})/X$ the quotient space of $\mathcal{P}_\alpha(\mathbb{K})$ obtained by collapsing the image of this section.

The stable homotopy groups of $\mathcal{P}_\alpha(\mathbb{K})/X$ by definition give the topological twisted K -homology groups $K_{ev/odd}^{\text{top}}(X, \alpha)$. (There are only two due to Bott periodicity of \mathbb{K} .) Thus we have

$$K_{ev}^{\text{top}}(X, \alpha) = \varinjlim_{k \rightarrow \infty} \pi_{2k}(\mathcal{P}_\alpha(BU(\infty))/X)$$

and

$$K_{odd}^{\text{top}}(X, \alpha) = \varinjlim_{k \rightarrow \infty} \pi_{2k+1}(\mathcal{P}_\alpha(BU(\infty))/X).$$

Here the direct limits are taken by the double suspension

$$\pi_{n+2k}(\mathcal{P}_\alpha(BU(\infty))/X) \longrightarrow \pi_{n+2k+2}(\mathcal{P}_\alpha(S^2 \wedge BU(\infty))/X)$$

and then followed by the standard map

$$\pi_{n+2k+2}(\mathcal{P}_\alpha(S^2 \wedge BU(\infty))/X) \xrightarrow{b \wedge 1} \pi_{n+2k+2}(\mathcal{P}_\alpha(BU(\infty) \wedge BU(\infty))/X)$$

$$\xrightarrow{m} \pi_{n+2k+2}(\mathcal{P}_\alpha(BU(\infty))/X)$$

where $b : \mathbb{R}^2 \rightarrow BU(\infty)$ represents the Bott generator in $K^0(\mathbb{R}^2) \cong \mathbb{Z}$, m is the base point preserving map inducing the ring structure on K -theory.

For a relative CW-complex (X, A) with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$, the relative version of topological twisted K -homology, denoted $K_{ev/odd}^{\text{top}}(X, A, \alpha)$, is defined to be $K_{ev/odd}^{\text{top}}(X/A, \alpha)$ where X/A is the quotient space of X obtained by collapsing A to a point. Then we have the following exact sequence

$$\begin{array}{ccccc} K_{odd}^{\text{top}}(X, A; \alpha) & \longrightarrow & K_{ev}^{\text{top}}(A, \alpha|_A) & \longrightarrow & K_{ev}^{\text{top}}(X, \alpha) \\ \uparrow & & & & \downarrow \\ K_{odd}^{\text{top}}(X, \alpha) & \longleftarrow & K_{odd}^{\text{top}}(A, \alpha|_A) & \longleftarrow & K_{ev}^{\text{top}}(X, A; \alpha) \end{array}$$

and the excision properties

$$K_{ev/odd}^{\text{top}}(X, B; \alpha) \cong K_{ev/odd}^{\text{top}}(A, A - B; \alpha|_A)$$

for any CW-triad $(X; A, B)$ with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$. A triple $(X; A, B)$ is a CW-triad if X is a CW-complex, and A, B are two subcomplexes of X such that $A \cup B = X$.

For the analytic twisted K -homology, recall that $\mathcal{P}_\alpha(\mathcal{K})$ is the associated bundle of compact operators on X . Analytic twisted K -homology, denoted by $K_{ev/odd}^{\text{an}}(X, \alpha)$, is defined to be

$$K_{ev/odd}^{\text{an}}(X, \alpha) := KK^{ev/odd}(C_0(X, \mathcal{P}_\alpha(\mathcal{K})), \mathbb{C}),$$

Kasparov's \mathbb{Z}_2 -graded K -homology of the C^* -algebra $C_0(X, \mathcal{P}_\alpha(\mathcal{K}))$.

For a relative CW-complex (X, A) with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$, the relative version of analytic twisted K -homology $K_{ev/odd}^{\text{an}}(X, A, \alpha)$ is defined to be $K_{ev/odd}^{\text{an}}(X - A, \alpha)$. Then we have the following exact sequence

$$\begin{array}{ccccc} K_{odd}^{\text{an}}(X, A; \alpha) & \longrightarrow & K_{ev}^{\text{an}}(A, \alpha|_A) & \longrightarrow & K_{ev}^{\text{an}}(X, \alpha) \\ \uparrow & & & & \downarrow \\ K_{odd}^{\text{an}}(X, \alpha) & \longleftarrow & K_{odd}^{\text{an}}(A, \alpha|_A) & \longleftarrow & K_{ev}^{\text{an}}(X, A; \alpha) \end{array}$$

and the excision properties

$$K_{ev/odd}^{\mathbf{an}}(X, B; \alpha) \cong K_{ev/odd}^{\mathbf{an}}(A, A - B; \alpha|_A)$$

for any CW-triad $(X; A, B)$ with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$.

THEOREM 3.1. (*Theorem 5.1 in [Wa]*) *There is a natural isomorphism*

$$\Phi : K_{ev/odd}^{\mathbf{top}}(X, \alpha) \longrightarrow K_{ev/odd}^{\mathbf{an}}(X, \alpha)$$

for any **smooth** manifold X with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$.

The proof of this theorem requires Poincaré duality between twisted K -theory and twisted K -homology (we describe this duality in the next theorem), and the isomorphism (Theorem 2.2) between topological twisted K -theory and analytic twisted K -theory.

Fix an isomorphism $\mathcal{H} \otimes \mathcal{H} \cong \mathcal{H}$ which induces a group homomorphism $U(\mathcal{H}) \times U(\mathcal{H}) \longrightarrow U(\mathcal{H})$ whose restriction to the center is the group multiplication on $U(1)$. So we have a group homomorphism

$$PU(\mathcal{H}) \times PU(\mathcal{H}) \longrightarrow PU(\mathcal{H})$$

which defines a continuous map, denoted m_* , of CW-complexes

$$BPU(\mathcal{H}) \times BPU(\mathcal{H}) \longrightarrow BPU(\mathcal{H}).$$

As $BPU(\mathcal{H})$ is identified as $K(\mathbb{Z}, 3)$, we may think of this as a continuous map taking $K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 3)$ to $K(\mathbb{Z}, 3)$, which can be used to define $\alpha + o_X$.

There are natural isomorphisms from twisted K -homology (topological resp. analytic) to twisted K -theory (topological resp. analytic) of a smooth manifold X where the twisting is shifted by

$$\alpha \mapsto \alpha + o_X$$

where $\tau : X \rightarrow BSO$ is the classifying map of the stable tangent space and $\alpha + o_X$ denotes the map $X \rightarrow K(\mathbb{Z}, 3)$, representing the class $[\alpha] + W_3(X)$ in $H^3(X, \mathbb{Z})$.

THEOREM 3.2. *Let X be a smooth manifold with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$. There exist isomorphisms*

$$K_{ev/odd}^{\mathbf{top}}(X, \alpha) \cong K_{\mathbf{top}}^{ev/odd}(X, \alpha + o_X)$$

and

$$K_{ev/odd}^{\mathbf{an}}(X, \alpha) \cong K_{\mathbf{an}}^{ev/odd}(X, \alpha + o_X)$$

with the degree shifted by $\dim X \pmod{2}$.

Analytic Poincaré duality was established in [EEK] and [Tu], and topological Poincaré duality was established in [Wa]. Theorem 3.1 and the exact sequences for a pair (X, A) imply the following corollary.

COROLLARY 3.3. *There is a natural isomorphism*

$$\Phi : K_{ev/odd}^{\mathbf{top}}(X, A, \alpha) \longrightarrow K_{ev/odd}^{\mathbf{an}}(X, A, \alpha)$$

for any **smooth** manifold X with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ and a closed submanifold $A \subset X$.

REMARK 3.4. In fact, Poincaré duality as in Theorem 3.2 holds for any compact Riemannian manifold W with boundary ∂W and a twisting $\alpha : W \rightarrow K(\mathbb{Z}, 3)$. This duality takes the following form

$$K_{ev/odd}^{\mathbf{top}}(W, \alpha) \cong K_{\mathbf{top}}^{ev/odd}(W, \partial W, \alpha + o_W)$$

and

$$K_{ev/odd}^{\text{an}}(W, \alpha) \cong K_{\text{an}}^{ev/odd}(X, \partial X, \alpha + o_W)$$

with the degree shifted by $\dim W \pmod{2}$. From this, we have a natural isomorphism ([BW])

$$\Phi : K_{ev/odd}^{\text{top}}(X, A, \alpha) \longrightarrow K_{ev/odd}^{\text{an}}(X, A, \alpha)$$

for any CW pair (X, A) with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ using the Five Lemma.

3.2. Geometric cycles and geometric twisted K -homology. Let X be a paracompact Hausdorff space and let $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ be a twisting over X .

DEFINITION 3.5. Given a smooth oriented manifold M with a classifying map ν of its stable normal bundle then we say that M is an α -twisted $Spin^c$ manifold over X if M is equipped with an α -twisted $Spin^c$ structure, that means, a continuous map $\iota : M \rightarrow X$ such that the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\nu} & \mathbf{BSO} \\ \downarrow \iota & \searrow \eta & \downarrow W_3 \\ X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3), \end{array}$$

commutes up to a fixed homotopy η from $W_3 \circ \nu$ and $\alpha \circ \iota$. Such an α -twisted $Spin^c$ manifold over X will be denoted by (M, ν, ι, η) .

PROPOSITION 3.6. M admits an α -twisted $Spin^c$ structure if and only if there is a continuous map $\iota : M \rightarrow X$ such that

$$\iota^*([\alpha]) + W_3(M) = 0.$$

If ι is an embedding, this is the anomaly cancellation condition obtained by Freed and Witten in [FreWit].

PROOF. This is clear. \square

A morphism between α -twisted $Spin^c$ manifolds $(M_1, \nu_1, \iota_1, \eta_1)$ and $(M_2, \nu_2, \iota_2, \eta_2)$ is a continuous map $f : M_1 \rightarrow M_2$ where the following diagram

$$(3.2) \quad \begin{array}{ccccc} M_1 & & \xrightarrow{\nu_1} & & \mathbf{BSO} \\ & \searrow f & & \searrow \nu_2 & \\ & & M_2 & & \\ & & \downarrow \iota_2 & \searrow \eta_2 & \downarrow W_3 \\ & & X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3) \end{array}$$

is a homotopy commutative diagram such that

- (1) ν_1 is homotopic to $\nu_2 \circ f$ through a continuous map $\nu : M_1 \times [0, 1] \rightarrow \mathbf{BSO}$;
- (2) $\iota_2 \circ f$ is homotopic to ι_1 through continuous map $\iota : M_1 \times [0, 1] \rightarrow X$;
- (3) the composition of homotopies $(\alpha \circ \iota) * (\eta_2 \circ (f \times Id)) * (W_3 \circ \nu)$ is homotopic to η_1 .

Two α -twisted $Spin^c$ manifolds $(M_1, \nu_1, \iota_1, \eta_1)$ and $(M_2, \nu_2, \iota_2, \eta_2)$ are called isomorphic if there exists a diffeomorphism $f : M_1 \rightarrow M_2$ such that the above holds. If the identity map on M induces an isomorphism between $(M, \nu_1, \iota_1, \eta_1)$ and $(M, \nu_2, \iota_2, \eta_2)$, then these two α -twisted $Spin^c$ structures are called equivalent.

Orientation reversal in the Grassmannian model defines an involution

$$r : \mathbf{BSO} \longrightarrow \mathbf{BSO}.$$

Choose a good cover $\{V_i\}$ of M and hence a trivialisation of the universal bundle over $\mathbf{BSO}(n)$ with transition functions

$$g_{ij} : V_i \cap V_j \longrightarrow SO(n).$$

Let $\tilde{g}_{ij} : V_i \cap V_j \longrightarrow Spin^c(n)$ be a lifting of g_{ij} . Then $\{c_{ijk}\}$, obtained from

$$\tilde{g}_{ij}\tilde{g}_{jk} = c_{ijk}\tilde{g}_{ik},$$

defines $[W_3] \in H^3(\mathbf{BSO}, \mathbb{Z})$. Let h be the diagonal matrix with the first $(n-1)$ diagonal entries 1 and the last entry -1 . Then $\{hg_{ij}h^{-1}\}$ are the transition functions for the universal bundle over $\mathbf{BSO}(n)$ with the opposite orientation. Note that $\{h\tilde{g}_{ij}h^{-1}\}$ is a lifting of $\{hg_{ij}h^{-1}\}$, which leaves $\{c_{ijk}\}$ unchanged. We have $[W_3] = [W_3 \circ r] \in H^3(\mathbf{BSO}, \mathbb{Z})$. Hence there is a homotopy connecting W_3 and $W_3 \circ r$. (It is unique up to homotopy as $H^2(\mathbf{BSO}, \mathbb{Z}) = 0$). Given an α -twisted $Spin^c$ manifold (M, ν, ι, η) , let $-M$ be the same manifold with the orientation reversed. Then the homotopy commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\nu} & \mathbf{BSO} & \xrightarrow{r} & \mathbf{BSO} \\ \downarrow \iota & \searrow \eta & \downarrow W_3 & \swarrow W_3 & \\ X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3) & & \end{array}$$

determines a unique equivalence class of α -twisted $Spin^c$ structure on $-M$, called the **opposite** α -twisted $Spin^c$ structure, simply denoted by $-(M, \nu, \iota, \eta)$.

DEFINITION 3.7. A geometric cycle for (X, α) is a quintuple $(M, \iota, \nu, \eta, [E])$ where $[E]$ is a K-class in $K^0(M)$ and M is a smooth closed manifold equipped with an α -twisted $Spin^c$ structure (M, ι, ν, η) .

Two geometric cycles $(M_1, \iota_1, \nu_1, \eta_1, [E_1])$ and $(M_2, \iota_2, \nu_2, \eta_2, [E_2])$ are isomorphic if there is an isomorphism $f : (M_1, \iota_1, \nu_1, \eta_1) \rightarrow (M_2, \iota_2, \nu_2, \eta_2)$, as α -twisted $Spin^c$ manifolds over X , such that $f_!([E_1]) = [E_2]$.

Let $\Gamma(X, \alpha)$ be the collection of all geometric cycles for (X, α) . We now impose an equivalence relation \sim on $\Gamma(X, \alpha)$, generated by the following three elementary relations:

(1) **Direct sum - disjoint union**

If $(M, \iota, \nu, \eta, [E_1])$ and $(M, \iota, \nu, \eta, [E_2])$ are two geometric cycles with the same α -twisted $Spin^c$ structure, then

$$(M, \iota, \nu, \eta, [E_1]) \cup (M, \iota, \nu, \eta, [E_2]) \sim (M, \iota, \nu, \eta, [E_1] + [E_2]).$$

(2) **Bordism**

Given two geometric cycles $(M_1, \iota_1, \nu_1, \eta_1, [E_1])$ and $(M_2, \iota_2, \nu_2, \eta_2, [E_2])$, if there exists a α -twisted $Spin^c$ manifold (W, ι, ν, η) and $[E] \in K^0(W)$ such that

$$\partial(W, \iota, \nu, \eta) = -(M_1, \iota_1, \nu_1, \eta_1) \cup (M_2, \iota_2, \nu_2, \eta_2)$$

and $\partial([E]) = [E_1] \cup [E_2]$. Here $-(M_1, \iota_1, \nu_1, \eta_1)$ denotes the manifold M_1 with the opposite α -twisted $Spin^c$ structure.

(3) $Spin^c$ vector bundle modification

Suppose we are given a geometric cycle $(M, \iota, \nu, \eta, [E])$ and a $Spin^c$ vector bundle V over M with even dimensional fibers. Denote by $\underline{\mathbb{R}}$ the trivial rank one real vector bundle. Choose a Riemannian metric on $V \oplus \underline{\mathbb{R}}$, let

$$\hat{M} = S(V \oplus \underline{\mathbb{R}})$$

be the sphere bundle of $V \oplus \underline{\mathbb{R}}$. Then the vertical tangent bundle $T^v(\hat{M})$ of \hat{M} admits a natural $Spin^c$ structure with an associated \mathbb{Z}_2 -graded spinor bundle $S_V^+ \oplus S_V^-$. Denote by $\rho : \hat{M} \rightarrow M$ the projection which is K -oriented. Then

$$(M, \iota, \nu, \eta, [E]) \sim (\hat{M}, \iota \circ \rho, \nu \circ \rho, \eta \circ \rho, [\rho^* E \otimes S_V^+]).$$

DEFINITION 3.8. Denote by $K_*^{\text{geo}}(X, \alpha) = \Gamma(X, \alpha) / \sim$ the geometric twisted K -homology. Addition is given by disjoint union - direct sum relation. Note that the equivalence relation \sim preserves the parity of the dimension of the underlying α -twisted $Spin^c$ manifold. Let $K_0^{\text{geo}}(X, \alpha)$ (resp. $K_1^{\text{geo}}(X, \alpha)$) the subgroup of $K_*^{\text{geo}}(X, \alpha)$ determined by all geometric cycles with even (resp. odd) dimensional α -twisted $Spin^c$ manifolds.

REMARK 3.9. (1) If M , in a geometric cycle $(M, \iota, \nu, \eta, [E])$ for (X, α) , is a compact manifold with boundary, then $[E]$ has to be a class in $K^0(M, \partial M)$.
 (2) If $f : X \rightarrow Y$ is a continuous map and $\alpha : Y \rightarrow K(\mathbb{Z}, 3)$ is a twisting, then there is a natural homomorphism of abelian groups

$$f_* : K_{\text{ev/odd}}^{\text{geo}}(X, \alpha \circ f) \longrightarrow K_{\text{ev/odd}}^{\text{geo}}(Y, \alpha)$$

sending $[M, \iota, \nu, \eta, E]$ to $[M, f \circ \iota, \nu, \eta, E]$.

- (3) Let A be a closed subspace of X , and α be a twisting on X . A relative geometric cycle for $(X, A; \alpha)$ is a quintuple $(M, \iota, \nu, \eta, [E])$ such that
- (a) M is a smooth manifold (possibly with boundary), equipped with an α -twisted $Spin^c$ structure (M, ι, ν, η) ;
 - (b) if M has a non-empty boundary, then $\iota(\partial M) \subset A$;
 - (c) $[E]$ is a K -class in $K^0(M)$ represented by a \mathbb{Z}_2 -graded vector bundle E over M , or a continuous map $M \rightarrow BU(\infty)$.

The relation \sim generated by disjoint union - direct sum, bordism and $Spin^c$ vector bundle modification is an equivalence relation. The collection of relative geometric cycles, modulo the equivalence relation is denoted by

$$K_{\text{ev/odd}}^{\text{geo}}(X, A; \alpha).$$

There exists a natural homomorphism, called the assembly map

$$\mu : K_{\text{ev/odd}}^{\text{geo}}(X, \alpha) \rightarrow K_{\text{ev/odd}}^{\text{an}}(X, \alpha)$$

whose definition (which we will now explain) requires a careful study of geometric cycles.

Given a geometric cycle $(M, \iota, \nu, \eta, [E])$, equip M with a Riemannian metric. Denote by $\mathbf{Cliff}(TM)$ the bundle of complex Clifford algebras of TM over M . The algebra of sections, $C(M, \mathbf{Cliff}(TM))$, is Morita equivalent to $C(M, \tau^* \mathbf{BSpin}^c(\mathcal{K}))$. Hence, we have a canonical isomorphism

$$K_{\text{ev/odd}}^{\text{an}}(M, W_3 \circ \tau) \cong KK^{\text{ev/odd}}(C(M, \mathbf{Cliff}(M)), \mathbb{C})$$

with the degree shift by $\dim M \pmod{2}$. Applying Kasparov's Poincaré duality (Cf. [Kas2])

$$KK^{\text{ev/odd}}(\mathbb{C}, C(M)) \cong KK^{\text{ev/odd}}(C(M, \mathbf{Cliff}(M)), \mathbb{C}),$$

we obtain a canonical isomorphism

$$PD : K^0(M) \cong K_{ev/odd}^{\text{an}}(M, o_M),$$

with the degree shift by $\dim M \pmod{2}$. The fundamental class $[M] \in K_{ev/odd}^{\text{an}}(M, o_M)$ is the Poincaré dual of the unit element in $K^0(M)$. Note that $[M] \in K_{ev}^{\text{an}}(M, o_M)$ if M is even dimensional and $[M] \in K_{odd}^{\text{an}}(M, o_M)$ if M is odd dimensional. The cap product

$$\cap : K_{ev/odd}^{\text{an}}(M, o_M) \otimes K^0(M) \longrightarrow K_{ev/odd}^{\text{an}}(M, o_M)$$

is defined by the Kasparov product. We remark that Poincaré duality is given by the cap product of the fundamental K -homology class $[M]$

$$[M] \cap : K^0(M) \cong K_{ev/odd}^{\text{an}}(M, o_M).$$

Choose an embedding $i_k : M \rightarrow \mathbb{R}^{n+k}$ and take the resulting normal bundle ν_M . The natural isomorphism

$$TM \oplus \nu_M \oplus \nu_M \cong \mathbb{R}^{n+k} \oplus \nu_M$$

and the canonical $Spin^c$ structure on $\nu_M \oplus \nu_M$ define a canonical homotopy between the orientation twisting o_M of TM and the orientation twisting o_{ν_M} of ν_M . This canonical homotopy defines an isomorphism

$$(3.3) \quad I_* : K_{ev/odd}^{\text{an}}(M, o_M) \cong K_{ev/odd}^{\text{an}}(M, o_{\nu_M}).$$

Given an α -twisted $Spin^c$ manifold (M, ν, ι, η) over X , the homotopy η induces an isomorphism $\nu^* \mathbf{BSpin}^c \cong \iota^* \mathcal{P}_\alpha$ as principal $K(\mathbb{Z}, 2)$ -bundles on M . Hence there is an isomorphism

$$\nu^* \mathbf{BSpin}^c(\mathcal{K}) \xrightarrow[\cong]{\eta^*} \iota^* \mathcal{P}_\alpha(\mathcal{K})$$

as bundles of C^* -algebras on M . This isomorphism determines a canonical isomorphism between the corresponding continuous trace C^* -algebras

$$C(M, \nu^* \mathbf{BSpin}^c(\mathcal{K})) \cong C(M, \iota^* \mathcal{P}_\alpha(\mathcal{K})).$$

Hence, we have a canonical isomorphism

$$(3.4) \quad \eta_* : K_{ev/odd}^{\text{an}}(M, o_{\nu_M}) \cong K_{ev/odd}^{\text{an}}(M, \alpha \circ \iota).$$

Now we can define the assembly map as

$$\mu(M, \iota, \nu, \eta, [E]) = \iota_* \circ \eta_* \circ I_*([M] \cap [E])$$

in $K_{ev/odd}^{\text{an}}(X, \alpha)$. Here ι_* is the natural push-forward map in analytic twisted K -homology.

THEOREM 3.10. (*Theorem 6.4 in [Wa]*) *The assembly map $\mu : K_{ev/odd}^{\text{geo}}(X, \alpha) \rightarrow K_{ev/odd}^{\text{an}}(X, \alpha)$ is an isomorphism for any smooth manifold X with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$.*

The proof follows by establishing the existence of a natural map $\Psi : K_{ev/odd}^{\text{top}}(X, \alpha) \rightarrow K_0^{\text{geo}}(X, \alpha)$ such that the following diagram

$$\begin{array}{ccc} & K_{ev/odd}^{\text{top}}(X, \alpha) & \\ \Psi \swarrow & & \searrow \Phi \\ K_{ev/odd}^{\text{geo}}(X, \alpha) & \xrightarrow{\mu} & K_{ev/odd}^{\text{an}}(X, \alpha) \end{array}$$

commutes. All the maps in the diagram are isomorphisms.

REMARK 3.11. This theorem is generalised in [BW] to the case of any CW pair (X, A) . That is, it is shown that the equivalence between the geometric twisted K -homology and the analytic twisted K -theory holds in this more general situation.

COROLLARY 3.12. $K_{ev/odd}^{\text{an}}(X, \alpha) \cong K_{ev/odd}^{\text{an}}(X, -\alpha)$.

PROOF. By the Brown representation theorem ([Swi]), there is a continuous map $\hat{i} : K(\mathbb{Z}, 3) \rightarrow K(\mathbb{Z}, 3)$ (unique up to homotopy as $H^2(K(\mathbb{Z}, 3), \mathbb{Z}) = 0$) such that

$$[\hat{i} \circ \alpha] = -[\alpha] \in H^3(X, \mathbb{Z})$$

for any map $\alpha : X \rightarrow K(\mathbb{Z}, 3)$. Then we have

$$[\hat{i} \circ W_3] = -[W_3] \in H^3(BSO, \mathbb{Z}).$$

As $[W_3]$ is 2-torsion, we know that $[\hat{i} \circ W_3] = -[W_3] = [W_3]$. Therefore, there is a homotopy η_0 connecting $\hat{i} \circ W_3$ and W_3 , that is, the following diagram is homotopy commutative

$$\begin{array}{ccc} BSO & \xrightarrow{W_3} & K(\mathbb{Z}, 3) \\ \downarrow W_3 & \searrow \eta_0 & \uparrow \hat{i} \\ K(\mathbb{Z}, 3) & \xrightarrow{\hat{i}} & K(\mathbb{Z}, 3) \end{array}$$

Note that the homotopy class of η_0 as a homotopy connecting W_3 and $\hat{i} \circ W_3$ is unique due to the fact that $H^2(BSO, \mathbb{Z}) = 0$.

Given an α -twisted $Spin^c$ manifold (M, ι, ν, η) , then the following homotopy commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\nu} & BSO & \xrightarrow{W_3} & K(\mathbb{Z}, 3) \\ \downarrow \iota & \searrow \eta & \downarrow W_3 & \searrow \eta_0 & \uparrow \hat{i} \\ X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3) & \xrightarrow{\hat{i}} & K(\mathbb{Z}, 3) \end{array}$$

defines a unique (due to $H^2(BSO, \mathbb{Z}) = 0$) equivalence class of $(-\alpha)$ -twisted $Spin^c$ structures. Here $-\alpha = \hat{i} \circ \alpha$. We denote by $\hat{i}(M, \iota, \nu, \eta)$ this $(-\alpha)$ -twisted $Spin^c$ manifold. Obviously,

$$\hat{i}(\hat{i}(M, \iota, \nu, \eta)) = (M, \iota, \nu, \eta).$$

The isomorphism $K_{ev/odd}^{\text{an}}(X, \alpha) \cong K_{ev/odd}^{\text{an}}(X, -\alpha)$ is induced by the involution \hat{i} on geometric cycles. \square

4. The Chern character in twisted K -theory

In this Section, we will review the Chern character map in twisted K -theory on smooth manifolds developed in [CMW] using gerbe connections and curvings. For the topological and analytic definitions, see [AS2] and [MatSte] respectively. Recently, Gomi and Terashima in [GoTe] gave another construction of a Chern character for twisted K -theory using a notion of connection on a finite-dimensional approximation of a twisted family of Fredholm operators developed by Gomi ([Gomi]).

4.1. Twisted Chern character. For a fibration $\pi^* : Y \rightarrow X$, let $Y^{[p]}$ denote the p th fibered product. There are projection maps $\pi_i : Y^{[p]} \rightarrow Y^{[p-1]}$ which omit the i th element for each $i = 1 \dots p$. These define a map

$$(4.1) \quad \delta : \Omega^q(Y^{[p-1]}) \rightarrow \Omega^q(Y^{[p]})$$

by

$$(4.2) \quad \delta(\omega) = \sum_{i=1}^p (-1)^i \pi_i^*(\omega).$$

Clearly $\delta^2 = 0$. In fact, the δ -cohomology of this complex vanishes identically, hence, the sequence

$$0 \longrightarrow \Omega^q(X) \xrightarrow{\pi^*} \Omega^q(Y) \cdots \xrightarrow{\delta} \Omega^q(Y^{[p-1]}) \xrightarrow{\delta} \Omega^q(Y^{[p]}) \longrightarrow \cdots$$

is exact.

Returning now to our particular example, a bundle gerbe connection on \mathcal{P}_α is a unitary connection θ on the principal $U(1)$ -bundle \mathcal{G}_α over $\mathcal{P}_\alpha^{[2]}$ which commutes with the bundle gerbe product. A bundle gerbe connection θ has curvature

$$F_\theta \in \Omega^2(\mathcal{P}_\alpha^{[2]})$$

satisfying $\delta(F_\theta) = 0$. There exists a two-form ω on \mathcal{P}_α such that

$$F_\theta = \pi_2^*(\omega) - \pi_1^*(\omega).$$

Such an ω is called a curving for the gerbe connection θ . The choice of a curving is not unique, the ambiguity in the choice is precisely the addition of the pull-back to \mathcal{P}_α of a two-form on X . Given a choice of curving ω , there is a unique closed three-form on β on X satisfying $d\omega = \pi^*\beta$. We denote by

$$\check{\alpha} = (\mathcal{G}_\alpha, \theta, \omega)$$

the lifting bundle gerbe \mathcal{G}_α with the connection θ and a curving ω . Moreover $H = \frac{\beta}{2\pi\sqrt{-1}}$ is a de Rham representative for the Dixmier-Douady class $[\alpha]$. We shall call $\check{\alpha}$ the **differential twisting**, as it is the twisting in differential twisted K -theory (Cf. [CMW]).

The following theorem is established in [CMW].

THEOREM 4.1. *Let X be a smooth manifold, $\pi : \mathcal{P}_\alpha \rightarrow X$ be a principal $PU(\mathcal{H})$ bundle over X whose classifying map is given by $\alpha : X \rightarrow K(\mathbb{Z}, 3)$. Let $\check{\alpha} = (\mathcal{G}_\alpha, \theta, \omega)$ be a bundle gerbe connection θ and a curving ω on the lifting bundle gerbe \mathcal{G}_α . There is a well-defined twisted Chern character*

$$Ch_{\check{\alpha}} : K^*(X, \alpha) \longrightarrow H^{ev/odd}(X, d - H).$$

Here the groups $H^{ev/odd}(X, d - H)$ are the twisted cohomology groups of the complex of differential forms on X with the coboundary operator given by $d - H$. The twisted Chern character is functorial under the pull-back. Moreover, given another differential twisting $\check{\alpha} + b = (\mathcal{G}_\alpha, \theta, \omega + \pi^*b)$ for a 2-form b on X ,

$$Ch_{\check{\alpha}+b} = Ch_{\check{\alpha}} \cdot \exp\left(\frac{b}{2\pi\sqrt{-1}}\right).$$

PROOF. Choose a good open cover $\{V_i\}$ of X such that $\mathcal{P}_\alpha \rightarrow X$ has trivializing sections ϕ_i over each V_i with transition functions $g_{ij} : V_i \cap V_j \rightarrow PU(\mathcal{H})$ satisfying $\phi_j = \phi_i g_{ij}$. Define $\{\sigma_{ijk}\}$ by $\hat{g}_{ij} \hat{g}_{jk} = \hat{g}_{ik} \sigma_{ijk}$ for a lift of g_{ij} to $\hat{g}_{ij} : V_i \cap V_j \rightarrow U(\mathcal{H})$. Note that the pair (ϕ_i, ϕ_j) defines a section of $\mathcal{P}_\alpha^{[2]}$ over $V_i \cap V_j$. The connection θ can be pulled back by (ϕ_i, ϕ_j) to define a 1-form A_{ij} on $V_i \cap V_j$ and the curving ω can be pulled-back by the ϕ_i to define two-forms B_i on V_i . Then the differential twisting defines the triple

$$(4.3) \quad \{(\sigma_{ijk}, A_{ij}, B_i)\}$$

which is a degree two smooth Deligne cocycle. Now we explain in some detail the twisted Chern characters in both the odd and even case following [CMW].

The even case: As a model for the K^0 classifying space, we choose **Fred**, the space of bounded self-adjoint Fredholm operators with essential spectrum $\{\pm 1\}$ and otherwise discrete spectra, with a grading operator Γ which anticommutes with the given family of Fredholm operators.

A twisted K -class in $K^0(X, \alpha)$ can be represented by $f : \mathcal{P}_\alpha \rightarrow \mathbf{Fred}$, a $PU(\mathcal{H})$ -equivariant family of Fredholm operators. We can select an open cover $\{V_i\}$ of X such that on each V_i there is a local section $\phi_i : V_i \rightarrow \mathcal{P}_\alpha$ and for each i the Fredholm operators $f(\phi_i(x))$, $x \in V_i$ have a gap in the spectrum at both $\pm \lambda_i \neq 0$. Then over V_i we have a finite rank vector bundle E_i defined by the spectral projections of the operators $f(\phi_i(x))$ corresponding to the interval $[-\lambda_i, \lambda_i]$.

Passing to a finer cover $\{U_i\}$ if necessary, we may assume that E_i is a trivial vector bundle over U_i of rank n_i . Choosing a trivialization of E_i gives a \mathbb{Z}_2 graded parametrix q_i (an inverse up to finite rank operators) of the family $f \circ \phi_i$. In the index zero sector the operator $q_i(x)^{-1}$ is defined as the direct sum of the restriction of $f(\phi_i(x))$ to the orthogonal complement of E_i in \mathcal{H} and an isomorphism between the vector bundles E_i^+ and E_i^- . Clearly then $f(\phi_i(x))q_i(x) = 1$ modulo rank n_i operators. In the case of nonzero index one defines a parametrix as a graded invertible operator q_i such that $f(\phi_i(x))q_i(x) = s_n$ modulo finite rank operators, with s_n a fixed Fredholm operator of index n equal to the index of $f(\phi_i(x))$.

On the overlap U_{ij} we have a pair of parametrices q_i and q_j of families of $f \circ \phi_i$ and $f \circ \phi_j$ respectively. These are related by an invertible operator f_{ij} which is of the form $1 +$ a finite rank operator,

$$\hat{g}_{ij} q_j(x) \hat{g}_{ij}^{-1} = q_i(x) f_{ij}(x).$$

The conjugation on the left hand side by \hat{g}_{ij} comes from the equivariance relation

$$f(\phi_j(x)) = f(\phi_i(x) g_{ij}(x)) = \hat{g}_{ij}(x)^{-1} f(\phi_i(x)) \hat{g}_{ij}(x).$$

The system $\{f_{ij}\}$ does not quite satisfy the Čech cocycle relation needed to define a principal bundle, because of the different local sections $\phi_i : U_i \rightarrow \mathcal{P}_\sigma$ involved. Instead, we have on U_{ijk}

$$\hat{g}_{jk} q_k \hat{g}_{jk}^{-1} = q_j f_{jk} = (\hat{g}_{ij}^{-1} q_i f_{ij} \hat{g}_{ij}) f_{jk} = \hat{g}_{jk} (\hat{g}_{ik}^{-1} q_i f_{ik} \hat{g}_{ik}) \hat{g}_{jk}^{-1}.$$

Using the relation $\hat{g}_{jk} \hat{g}_{ik}^{-1} = \sigma_{ijk} \hat{g}_{ij}^{-1}$, we get

$$\hat{g}_{jk} (\hat{g}_{ik}^{-1} q_i f_{ik} \hat{g}_{ik}) \hat{g}_{jk}^{-1} = \hat{g}_{ij}^{-1} q_i f_{ik} \hat{g}_{ij}$$

multiplying the last equation from right by \hat{g}_{ij}^{-1} and from the left by $q_i^{-1} \hat{g}_{ij}$ one gets the twisted cocycle relation

$$f_{ij} (\hat{g}_{ij} f_{jk} \hat{g}_{ij}^{-1}) = f_{ik},$$

which is independent of the choice of the lifting \hat{g}_{ij} . For simplicity, we will just write the above twisted cocycle relation as

$$(4.4) \quad f_{ij}(g_{ij}f_{jk}g_{ij}^{-1}) = f_{ik}.$$

This twisted cocycle relation (4.4) actually defines an untwisted cocycle relation for $\{(g_{ij}, f_{ij})\}$ in the twisted product

$$\mathfrak{G} = PU(\mathcal{H}) \rtimes GL(\infty),$$

where the group $PU(\mathcal{H})$ acts on the group $GL(\infty)$ of invertible $1+$ finite rank operators by conjugation. Thus the product in \mathfrak{G} is given by

$$(g, f) \cdot (g', f') = (gg', f(gf'g^{-1})).$$

The cocycle relation for the pairs $\{(g_{ij}, f_{ij})\}$ then encodes both the cocycle relation for the transition functions $\{g_{ij}\}$ of the $PU(\mathcal{H})$ bundle over X and the twisted cocycle relation (4.4). In summary, this cocycle $\{(g_{ij}, f_{ij})\}$ defines a principal \mathfrak{G} bundle over X .

The classifying space $B\mathfrak{G}$ is a fiber bundle over $K(\mathbb{Z}, 3)$. The fiber at each point in $K(\mathbb{Z}, 3)$ is homeomorphic (but not canonically so) to the space **Fred** of graded Fredholm operators; to set up the isomorphism one needs a choice of element in each fiber. Given a principal $PU(\mathcal{H})$ -bundle \mathcal{P}_α over X defined by $\alpha : X \rightarrow K(\mathbb{Z}, 3)$, the even twisted K -theory $K^0(X, \mathcal{P}_\alpha)$ is the set of homotopy classes of maps $X \rightarrow B\mathfrak{G}$ covering the map α .

Next we construct the twisted Chern character from a connection ∇ on a principal \mathfrak{G} bundle over X associated to the cocycle $\{(g_{ij}, f_{ij})\}$. Locally, on a good open cover $\{U_i\}$ of X we can lift the connection to a connection taking values in the Lie algebra $\hat{\mathfrak{g}}$ of the central extension $U(H) \times GL(\infty)$ of \mathfrak{G} . Denote by \hat{F}_∇ the curvature of this connection. On the overlaps $\{U_{ij}\}$ the curvature satisfies a twisted relation

$$\hat{F}_{\nabla, j} = Ad_{(g_{ij}, f_{ij})^{-1}} \hat{F}_{\nabla, i} + g_{ij}^* c,$$

where c is the curvature of the canonical connection θ on the principal $U(1)$ -bundle $U(\mathcal{H}) \rightarrow PU(\mathcal{H})$.

Since the Lie algebra $\mathfrak{u}(\infty) \oplus \mathbb{C}$ is an ideal in the Lie algebra of $U(\mathcal{H}) \rtimes GL(\infty)$, the projection $F'_{\nabla, i}$ of the curvature $\hat{F}_{\nabla, i}$ onto this subalgebra transforms in the same way as \hat{F} under change of local trivialization. It follows that for a $PU(\mathcal{H})$ -equivariant map $f : \mathcal{P}_\alpha \rightarrow \mathbf{Fred}$, we can define a twisted Chern character form of f as

$$(4.5) \quad ch_{\tilde{\alpha}}(f, \nabla) = e^{B_i \text{tr}} e^{F'_{\nabla, i}/2\pi i},$$

over V_i . Here the trace is well-defined on $\mathfrak{gl}(\infty)$ and on the center \mathbb{C} it is defined as the coefficient of the unit operator. Note that $ch_{\tilde{\alpha}}(f, \nabla)$ is globally defined and $(d - H)$ -closed

$$(d - H)ch_{\tilde{\alpha}}(f, \nabla) = 0,$$

and depends on the differential twisting

$$\tilde{\alpha} = (\mathcal{G}_\alpha, \theta, \omega).$$

Let f_0 and f_1 be homotopic, and ∇_0 and ∇_1 be two connections on the principal \mathfrak{G} bundle over X , we have a Chern-Simons type form

$$CS((f_0, \nabla_0), (f_1, \nabla_1)),$$

well-defined modulo $(d - H)$ -exact forms, such that

$$(4.6) \quad ch_{\tilde{\alpha}}(f_1, \nabla_1) - ch_{\tilde{\alpha}}(f_0, \nabla_0) = (d - H)CS((f_0, \nabla_0), (f_1, \nabla_1)).$$

The proof follows directly from the local computation using (4.5). Hence, the $(d - H)$ -cohomology class of $ch_{\check{\alpha}}(f, \nabla)$ does not depend choices of a connection ∇ on the principal \mathfrak{G} bundle over X , and depends only on the homotopy class of f . We denote the $(d - H)$ -cohomology class of $ch_{\check{\alpha}}(f, \nabla)$ by $Ch_{\check{\alpha}}([f_1])$ which is a natural homomorphism

$$Ch_{\check{\alpha}} : K^0(X, \alpha) \longrightarrow H^{ev}(X, d - H).$$

From (4.5), we have

$$Ch_{\check{\alpha}+b} = Ch_{\check{\alpha}} \cdot \exp\left(\frac{b}{2\pi\sqrt{-1}}\right).$$

for a differential twisting $\check{\alpha} + b = (\mathcal{G}_{\alpha}, \theta, \omega + \pi^*b)$.

The odd case: The odd case is a little easier. First, as a model for the K^1 classifying space, we choose $U(\infty) = \varinjlim_n U(n)$, the stabilized unitary group. Let Θ be the universal odd character form on $U(\infty)$ defined by the canonical left invariant $u(\infty)$ -valued form on $U(\infty)$.

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a polarized Hilbert space and let $U_{res} = U_{res}(\mathcal{H})$ denotes the group of unitary operators in \mathcal{H} with Hilbert-Schmidt off-diagonal blocks. The conjugation action of $U(\mathcal{H}_+) \times U(\mathcal{H}_-)$ on U_{res} defines an action of $PU_0(\mathcal{H}) = P(U(\mathcal{H}_+) \times U(\mathcal{H}_-))$ on U_{res} . Note that the classifying space of U_{res} is $U(\infty)$.

Define

$$\mathfrak{H} = PU_0(\mathcal{H}) \rtimes U_{res}.$$

Then given a principal $PU_0(\mathcal{H})$ -bundle \mathcal{P}_{α} over X defined by $\alpha : X \rightarrow K(\mathbb{Z}, 3)$, the odd twisted K -theory $K^1(X, \alpha)$ is the set of homotopy classes of maps $X \rightarrow B\mathfrak{H}$ covering the map α . These are represented by $PU_0(\mathcal{H})$ -equivariant maps $f : \mathcal{P}_{\alpha} \rightarrow U(\infty)$. With respect to trivializing sections ϕ_i over each V_i . Then

$$e^{B_i}(f \circ \phi_i)^* \Theta$$

is a globally defined and $(d - H)$ -closed differential form on X . This defines the odd version of the twisted Chern character

$$Ch_{\check{\alpha}} : K^1(X, \alpha) \longrightarrow H^{odd}(X, d - H).$$

□

4.2. Differential twisted K -theory. Recall that the Bockstein exact sequence in complex K -theory for any finite CW complex:

$$(4.7) \quad \begin{array}{ccccc} K^0(X) & \xrightarrow{ch} & H^{ev}(X, \mathbb{R}) & \longrightarrow & K_{\mathbb{R}/\mathbb{Z}}^0(X) \\ \uparrow & & & & \downarrow \\ K_{\mathbb{R}/\mathbb{Z}}^1(X) & \longleftarrow & H^{odd}(X, \mathbb{R}) & \xleftarrow{ch} & K^1(X) \end{array}$$

where $K_{\mathbb{R}/\mathbb{Z}}^*(X)$ is K -theory with \mathbb{R}/\mathbb{Z} -coefficients as in [Kar1] and [Ba].

Analogously, in twisted K -theory, given a smooth manifold X with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$, upon a choice of a differential twisting

$$\check{\alpha} = (\mathcal{G}_{\alpha}, \theta, \omega)$$

lifting α , we have the corresponding Bockstein exact sequence in twisted K -theory

$$(4.8) \quad \begin{array}{ccccc} K^0(X, \alpha) & \xrightarrow{Ch_{\tilde{\alpha}}} & H^{ev}(X, d - H) & \longrightarrow & K_{\mathbb{R}/\mathbb{Z}}^0(X, \alpha) \\ \uparrow & & & & \downarrow \\ K_{\mathbb{R}/\mathbb{Z}}^1(X, \alpha) & \longleftarrow & H^{odd}(X, d - H) & \xleftarrow{Ch_{\tilde{\alpha}}} & K^1(X, \alpha) \end{array}$$

Here $K_{\mathbb{R}/\mathbb{Z}}^0(X, \alpha)$ and $K_{\mathbb{R}/\mathbb{Z}}^1(X, \alpha)$ are subgroups of differential twisted K -theory, respectively $\check{K}^0(X, \tilde{\alpha})$ and $\check{K}^1(X, \tilde{\alpha})$ (see [CMW] for the detailed construction). Here we give another equivalent construction of differential twisted K -theory.

Fix a choice of a connection ∇ on a principal \mathfrak{G} bundle over X . Then $\check{K}^0(X, \tilde{\alpha})$ is the abelian group generated by pairs

$$\{(f, \eta)\},$$

modulo an equivalence relation, where $f : \mathcal{P}_{\alpha} \rightarrow \mathbf{Fred}$ is a $PU(\mathcal{H})$ -equivariant map and η is an odd differential form modulo $(d - H)$ -exact forms. Two pairs (f_0, η_0) and (f_1, η_1) are called equivalent if and only if

$$\eta_1 - \eta_0 = CS((f_1, \nabla), (f_0, \nabla)).$$

The differential Chern character form of f is given by

$$ch_{\tilde{\alpha}}(f, \nabla) - (d - H)\eta$$

which defines a homomorphism

$$ch_{\tilde{\alpha}} : \check{K}^0(X, \tilde{\alpha}) \longrightarrow \Omega_0^{ev}(X, d - H),$$

where $\Omega_0^{ev}(X, d - H)$ is the image of $ch_{\tilde{\alpha}} : \check{K}^0(X, \tilde{\alpha}) \rightarrow \Omega^{ev}(X)$. The kernel of $ch_{\tilde{\alpha}}$ is isomorphic to $K_{\mathbb{R}/\mathbb{Z}}^1(X, \alpha)$.

Similarly, we define the odd differential twisted K -theory $\check{K}^1(X, \tilde{\alpha})$ with the differential Chern character form homomorphism

$$ch_{\tilde{\alpha}} : \check{K}^1(X, \tilde{\alpha}) \longrightarrow \Omega_0^{odd}(X, d - H).$$

The kernel of $ch_{\tilde{\alpha}}$ is isomorphic to $K_{\mathbb{R}/\mathbb{Z}}^0(X, \alpha)$. The following commutative diagrams were established in [CMW] relating differential twisted K -theory with twisted K -theory

and with the diagram (4.8)

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 H^{odd}(X, d-H) & \longrightarrow & K_{\mathbb{R}/\mathbb{Z}}^1(X, \check{\sigma}) & & & & \\
 \downarrow & & \downarrow & \searrow & & & \\
 0 \rightarrow \frac{\Omega^{odd}(X)}{\Omega_0^{odd}(X, d-H)} \longrightarrow & \check{K}^0(X, \check{\sigma}) & \longrightarrow & K^0(X, \sigma) & \rightarrow & 0 & \\
 & \downarrow ch_{\check{\sigma}} & & \downarrow Ch_{\check{\sigma}} & & & \\
 & \Omega_0^{ev}(X, d-H) & \longrightarrow & H^{ev}(X, d-H) & & & \\
 & \downarrow & & & & & \\
 & 0 & & & & &
 \end{array}$$

$\swarrow d-H$ from $\frac{\Omega^{odd}(X)}{\Omega_0^{odd}(X, d-H)}$ to $\Omega_0^{ev}(X, d-H)$

and

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 H^{ev}(X, d-H) & \longrightarrow & K_{\mathbb{R}/\mathbb{Z}}^0(X, \check{\sigma}) & & & & \\
 \downarrow & & \downarrow & \searrow & & & \\
 0 \rightarrow \frac{\Omega^{ev}(X)}{\Omega_0^{ev}(X, d-H)} \longrightarrow & \check{K}^1(X, \check{\sigma}) & \longrightarrow & K^1(X, \sigma) & \rightarrow & 0 & \\
 & \downarrow ch_{\check{\sigma}} & & \downarrow Ch_{\check{\sigma}} & & & \\
 & \Omega_0^{odd}(X, d-H) & \longrightarrow & H^{odd}(X, d-H) & & & \\
 & \downarrow & & & & & \\
 & 0 & & & & &
 \end{array}$$

$\swarrow d-H$ from $\frac{\Omega^{ev}(X)}{\Omega_0^{ev}(X, d-H)}$ to $\Omega_0^{odd}(X, d-H)$

with exact horizontal and vertical sequences, and exact upper-right and exact lower-left 4-term sequences. We expect that these two commutative diagrams uniquely characterize differential twisted K -theory.

4.3. Twisted Chern character for torsion twistings. In this paper, we will only use the twisted Chern character for a torsion twisting and in this case we will give an explicit construction. Let E be a real oriented vector bundle of rank $2k$ over X with its orientation twisting denoted by

$$o(E) : X \rightarrow K(\mathbb{Z}, 3).$$

The associated lifting bundle $\mathcal{G}_{o(E)}$ has a canonical reduction to the $Spin^c$ bundle gerbe $\mathcal{G}_{W_3(E)}$.

Choose a local trivialization of E over a good open cover $\{V_i\}$ of X . Then the transition functions

$$g_{ij} : V_i \cap V_j \longrightarrow SO(2k).$$

define an element in $H^1(X, \underline{SO}(2k))$ whose image under the Bockstein exact sequence

$$H^1(X, \underline{Spin}(2k)) \rightarrow H^1(X, \underline{SO}(2k)) \rightarrow H^2(X, \mathbb{Z}_2)$$

is the second Stieffel-Whitney class $w_2(E)$ of E . Denote the differential twisting by

$$\check{w}_2(E) = (\mathcal{G}_{W_3(E)}, \theta, 0),$$

the $Spin^c$ bundle gerbe $\mathcal{G}_{W_3(E)}$ with a flat connection θ and a trivial curving. With respect to a good cover $\{V_i\}$ of X the differential twisting $\check{w}_2(E)$ defines a Deligne cocycle $\{(\alpha_{ijk}, 0, 0)\}$ with trivial local B -fields, here $\alpha_{ijk} = \hat{g}_{ij}\hat{g}_{jk}\hat{g}_{ki}$ where $\hat{g}_{ij} : U_{ij} \rightarrow Spin^c(2n)$ is a lift of g_{ij} .

By Proposition 2.4, a twisted K-class in $K^0(X, o(E))$ can be represented by a Clifford bundle, denoted \mathcal{E} . Equip \mathcal{E} with a Clifford connection, and E with a $SO(2k)$ -connection. Locally, over each V_i we let $\mathcal{E}|_{V_i} \cong S_i \otimes \mathcal{E}_i$ where S_i is the local fundamental spinor bundle associated to $E|_{V_i}$ with the standard Clifford action of $\mathbf{Cliff}(E|_{V_i})$ obtained from the fundamental representation of $Spin(2k)$. Then \mathcal{E}_i is a complex vector bundle over V_i with a connection ∇_i such that on $V_i \cap V_j$

$$Ch(\mathcal{E}_i, \nabla_i) = Ch(\mathcal{E}_j, \nabla_j).$$

Hence, the twisted Chern character

$$Ch_{\check{w}_2(E)} : K^0(X, o(E)) \longrightarrow H^{ev}(X)$$

is given by $[\mathcal{E}] \mapsto \{[ch(\mathcal{E}_i, \nabla_i)] = ch(\mathcal{E}_i)\}$. The proof of the following proposition is straightforward.

PROPOSITION 4.2. *The twisted Chern character satisfies the following identities*

- (1) $Ch_{\check{w}_2(E_1 \oplus E_2)}([\mathcal{E}_1 \oplus \mathcal{E}_2]) = Ch_{\check{w}_2(E_1)}([\mathcal{E}_1]) + Ch_{\check{w}_2(E_2)}([\mathcal{E}_2]).$
- (2) $Ch_{\check{w}_2(E_1 \otimes E_2)}([\mathcal{E}_1 \otimes \mathcal{E}_2]) = Ch_{\check{w}_2(E_1)}([\mathcal{E}_1])Ch_{\check{w}_2(E_2)}([\mathcal{E}_2]).$

In the case that E has a $Spin^c$ structure whose determinant line bundle is L , there is a canonical isomorphism

$$K^0(X) \longrightarrow K^0(X, o(E)),$$

given by $[V] \mapsto [V \otimes S_E]$ where S_E is the associated spinor bundle of E . Then we have

$$Ch_{\check{w}_2(E)}([V \otimes S_E]) = e^{\frac{c_1(L)}{2}} ch([V]),$$

where $ch([V])$ is the ordinary Chern character of $[V] \in K^0(X)$.

In particular, when X is an even dimensional Riemannian manifold, and TX is equipped with the Levi-Civita connection, under the identification of $K^0(X, o(E))$ with the Grothendieck group of Clifford modules. Then

$$(4.9) \quad Ch_{\check{w}_2(X)}([\mathcal{E}]) = ch(\mathcal{E}/S)$$

where $ch(\mathcal{E}/S)$ is the relative Chern character of the Clifford module \mathcal{E} constructed in Section 4.1 of [BGV].

5. Thom classes and Riemann-Roch formula in twisted K -theory

5.1. The Thom class. Given any oriented real vector bundle $\pi : E \rightarrow X$ of rank $2k$, E admits a $Spin^c$ structure if its classifying map $\tau : X \rightarrow BSO(2k)$ admits a lift $\tilde{\tau}$

$$\begin{array}{ccc} & & \mathbf{BSpin}^c \\ & \nearrow \tilde{\tau} & \downarrow \\ X & \xrightarrow{\tau} & \mathbf{BSO}(2k). \end{array}$$

As $\mathbf{BSpin}^c \rightarrow \mathbf{BSO}(2k)$ is a $BU(1)$ -principal bundle with the classifying map given by

$$W_3 : \mathbf{BSO}(2k) \rightarrow K(\mathbb{Z}, 3),$$

E admits a $Spin^c$ structure if $W_3 \circ \tau : X \rightarrow K(\mathbb{Z}, 3)$ is null homotopic, and a choice of null homotopy determines a $Spin^c$ structure on E . Associated to a $Spin^c$ structure \mathfrak{s} on E , there is canonical K -theoretical Thom class

$$U_E^{\mathfrak{s}} = [\pi^* S^+, \pi^* S^-, cl] \in K_{cv}^0(E)$$

in the K -theory of E with vertical compact supports. Here S^+ and S^- are the positive and negative spinor bundle over X defined by the $Spin^c$ structure on E , and cl is the bundle map $\pi^* S^+ \rightarrow \pi^* S^-$ given by the Clifford action E on S^{\pm} .

- REMARK 5.1. (1) The restriction of $U_E^{\mathfrak{s}}$ to each fiber is a generator of $K^0(R^{2k})$, so a $Spin^c$ structure on E is equivalent to a K -orientation on E . Note that Thom classes and K -orientation are functorial under pull-backs of $Spin^c$ vector bundles.
- (2) Let $\mathfrak{s} \otimes L$ be another $Spin^c$ structure on E which differs from \mathfrak{s} by a complex line bundle $p : L \rightarrow X$, then

$$U_E^{\mathfrak{s}'} = U_E^{\mathfrak{s}} \cdot p^*([L]).$$

- (3) Let (E_1, \mathfrak{s}_1) and (E_2, \mathfrak{s}_2) be two $Spin^c$ vector bundles over X , p_1 and p_2 be the projections from $E_1 \oplus E_2$ to E_1 and E_2 respectively, then

$$U_{E_1 \oplus E_2}^{\mathfrak{s}_1 \oplus \mathfrak{s}_2} = p_1^*(U_{E_1}^{\mathfrak{s}_1}) \cdot p_2^*(U_{E_2}^{\mathfrak{s}_2}) \in K_{cv}^0(E_1 \oplus E_2).$$

- (4) The Thom isomorphism in K -theory for a $Spin^c$ vector bundle $\pi : E \rightarrow X$ of rank $2k$ is given by

$$\begin{array}{ccc} \Phi_E^K : & K^0(X) & \longrightarrow K^0(E) \\ & a & \longmapsto \pi^*(a)U_E^{\mathfrak{s}}. \end{array}$$

Here for locally compact spaces, we shall consider only K -theory with compact supports. When X is compact, $U_E^{\mathfrak{s}} \in K^0(TX)$ and the Thom isomorphism Φ_X^K is the inverse of the push-forward map

$$\pi_! : K^0(TX) \rightarrow K^0(X)$$

associated to the K -orientation of π defined the $Spin^c$ structure on E .

If an oriented vector bundle E of even rank over X does not admit a $Spin^c$ structure, $W_3 \circ \tau : X \rightarrow K(\mathbb{Z}, 3)$ is not null homotopic. Thus, $W_3 \circ \tau$ defines a twisting on X for K -theory, called the orientation twisting o_E . In this Section we will define a canonical Thom class

$$U_E \in K^0(E, \pi^* o_E)$$

such that $a \mapsto \pi^*(a) \cup U_E$ defines the Thom isomorphism $K^0(X, o_E) \cong K^0(E)$. In fact, $a \mapsto \pi^*(a) \cup U_E$ defines the Thom isomorphism (Cf. [CW1])

$$K^0(X, \alpha + o_E) \cong K^0(E, \alpha \circ \pi)$$

for any twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$.

Choose a good open cover $\{V_i\}$ of X such that $E_i = E|_{V_i}$ is trivialized by an isomorphism

$$E_i \cong V_i \times \mathbb{R}^{2n}.$$

This defines a canonical $Spin^c$ structure \mathfrak{s}_i on each E_i . Denote by $U_{E_i}^{\mathfrak{s}_i}$ the associated Thom class of (E_i, \mathfrak{s}_i) . Then we have

$$U_{E_j}^{\mathfrak{s}_j} = U_{E_i}^{\mathfrak{s}_i} \pi_{ij}^*([L_{ij}]) \in K_{cv}^0(E_{ij})$$

where L_{ij} is the difference line bundle over $V_{ij} = V_i \cap V_j$ defined by $\mathfrak{s}_j = \mathfrak{s}_i \otimes L_{ij}$ on $E_{ij} = E|_{V_{ij}}$. Recall that these local line bundles $\{L_{ij}\}$ define a bundle gerbe [Mur] associated to the twisting $o_E = W_3 \circ \tau : X \rightarrow K(\mathbb{Z}, 3)$ and a locally trivializing cover $\{V_i\}$. By the definition of twisted K -theory, $\{U_{E_i}^{\mathfrak{s}_i}\}$ defines a twisted K -theory class of E with compact vertical supports and twisting given by

$$\pi^*(o_E) = o_E \circ \pi : E \rightarrow K(\mathbb{Z}, 3).$$

We denote this canonical twisted K -theory class by

$$U_E \in K_{cv}^0(E, \pi^*(o_E)).$$

When X is compact, then $U_E \in K^0(E, \pi^*(o_E))$. One can easily show that the Thom class U_E does not depend on the choice of the trivializing cover.

Now we can list the properties of the Thom class in twisted K -theory.

PROPOSITION 5.2. (1) *If E is equipped with a $Spin^c$ structure \mathfrak{s} , then \mathfrak{s} defines a canonical isomorphism*

$$\phi_{\mathfrak{s}} : K_{cv}^0(E, \pi^*(o_E)) \longrightarrow K_{cv}^0(E)$$

such that $\phi_{\mathfrak{s}}(U_E) = U_E^{\mathfrak{s}}$.

(2) *Let $f : X \rightarrow Y$ be a continuous map and E be an oriented vector bundle of even rank over Y , then*

$$U_{f^*E} = f^*(U_E).$$

(3) *Let E_1 and E_2 be two oriented vector bundles of even rank over X , p_1 and p_2 be the projections from $E_1 \oplus E_2$ to E_1 and E_2 respectively, that is, we have the diagram*

$$\begin{array}{ccc} E_1 \oplus E_2 & \xrightarrow{p_2} & E_2 \\ \downarrow p_1 & & \downarrow \pi_2 \\ E_1 & \xrightarrow{\pi_1} & X, \end{array}$$

then

$$U_{E_1 \oplus E_2} = p_1^*(U_{E_1}) \cdot p_2^*(U_{E_2}).$$

(4) *Let $\pi : E \rightarrow X$ be an oriented vector bundle of even rank over a compact space X , the Thom isomorphism in twisted K -theory ([CW1])*

$$\Phi_E^K : K^0(X, \alpha + o_E) \cong K^0(E, \pi^*(\alpha))$$

is given by $a \mapsto \pi^*(a) \cdot U_E$. Moreover, the push-forward map in twisted K -theory ([CW1])

$$\pi_! : K^0(E, \pi^*(\alpha)) \longrightarrow K^0(X, \alpha + o_E)$$

satisfies $\pi_!(\pi^*(a) \cdot U_E) = a$.

PROOF. (1) The $Spin^c$ structure \mathfrak{s} defines canonical isomorphism

$$\phi_{\mathfrak{s}} : K_{cv}^0(E, \pi^*(o_E)) \longrightarrow K_{cv}^0(E)$$

as follows. Given a trivializing cover $\{V_i\}$ and the canonical $Spin^c$ structure \mathfrak{s}_i on $E_i = E|_{V_i}$, we have

$$\mathfrak{s}|_{E_i} = \mathfrak{s}_i \otimes L_i$$

for a complex line bundle $\pi_i : L_i \rightarrow V_i$. This implies $U_{E_i}^{\mathfrak{s}} = U_{E_i}^{\mathfrak{s}_i} \pi_i^*([L_i])$. Note that $L_{ij} = L_i \otimes L_j^*$.

Any twisted K -class a in $K_{cv}^0(E, \pi^*(o_E))$ is given by a local K -class a_i with compact vertical support such that $a_j = a_i \pi_{ij}^*([L_{ij}])$, then

$$a_i \pi_i^*([L_i]) = a_j \pi_j^*([L_j])$$

in $K_{cv}^0(E|_{V_{ij}})$. This defines the homomorphism ϕ , which is obviously an isomorphism sending U_E to $U_E^{\mathfrak{s}}$.

(2) Choose a good open cover $\{V_i\}$ of Y . By definition, the Thom class U_E is defined by $\{U_{E_i}^{\mathfrak{s}_i}\}$ with

$$U_{E_j}^{\mathfrak{s}_j} = U_{E_i}^{\mathfrak{s}_i} \pi_{ij}^*([L_{ij}]).$$

Then $\{f^{-1}(V_i)\}$ is an open cover of X , and $(f^*E)|_{f^{-1}(V_i)} = f^*E_i$ is trivialized with the canonical $Spin^c$ structure $f^*\mathfrak{s}_i$, thus

$$U_{f^*E_i}^{f^*\mathfrak{s}_i} = f^*U_{E_i}^{\mathfrak{s}_i}.$$

This gives $U_{f^*E} = f^*(U_E)$.

- (3) The proof is similar to the proof of (2).
 (4) From ([CW1]), we know that the Thom isomorphism and the push-forward map in twisted K -theory are both homomorphisms of $K^0(X, \alpha)$ -modules. There exists an oriented real vector bundle F of even rank such that

$$E \oplus F = X \times \mathbb{R}^{2m}$$

for some $m \in \mathbb{N}$. Thus, we have

$$\begin{array}{ccc} E & \xrightarrow{i} & X \times \mathbb{R}^{2m} \\ & \searrow \pi & \downarrow p \\ & & X \end{array}$$

From the construction of the push-forward map in ([CW1]), we see that

$$\pi_!(U_E) = p_! \circ i_!(U_E) = p_!(U_{E \oplus F}) = 1.$$

As the Thom isomorphism and the push-forward map in twisted K -theory are both homomorphisms of $K^0(X, \alpha)$ -modules, we get $\pi_!(\pi^*(a) \cdot U_E) = a$.

Note that the Thom isomorphism is inverse to the push-forward map $\pi_!$, hence, the Thom isomorphism in twisted K -theory

$$K^0(X, \alpha + o_E) \cong K^0(E, \pi^*(\alpha))$$

is given by $a \mapsto \pi^*(a) \cdot U_E$.

□

5.2. Twisted Riemann-Roch. By an application of the Thom class and Thom isomorphism in twisted K -theory, we will now give a direct proof of a special case of the Riemann-Roch theorem for twisted K -theory. With some notational changes, the argument can be applied to establish the general Riemann-Roch theorem in twisted K -theory. Denote by o_X and o_Y the orientation twistings associated to the tangent bundles $\pi_X : TX \rightarrow X$ and $\pi_Y : TY \rightarrow Y$ respectively.

THEOREM 5.3. *Given a smooth map $f : X \rightarrow Y$ between oriented manifolds, assume that $\dim Y - \dim X = 0 \pmod{2}$. Then the Riemann-Roch formula is given by*

$$Ch_{\tilde{w}_2(Y)}(f_!^K(a))\hat{A}(Y) = f_*^H(Ch_{\tilde{w}_2(X)}(a)\hat{A}(X)).$$

for any $a \in K^0(X, o_X)$. Here $\hat{A}(X)$ and $\hat{A}(Y)$ are the A -hat classes of X and Y respectively.

PROOF. For simplicity, assume that both X and Y are of even dimension, say $2m$ and $2n$ respectively, equipped with a Riemannian metric. We will consider Chern character defects in each of the following three squares

$$(5.1) \quad \begin{array}{ccccccc} K^0(X, o_X) & \xrightarrow[\cong]{\Phi_{TX}^K} & K_c^0(TX) & \xrightarrow{(df)_!^K} & K_c^0(TY) & \xrightarrow[\cong]{(\Phi_{TY}^K)^{-1}} & K^0(Y, o_Y) \\ Ch_{\tilde{w}_2(X)} \downarrow & & \downarrow Ch & & \downarrow Ch & & \downarrow Ch_{\tilde{w}_2(Y)} \\ H^{ev}(X) & \xrightarrow[\cong]{\Phi_{TX}^H} & H_c^{ev}(TX) & \xrightarrow{(df)_*^H} & H_c^{ev}(TY) & \xrightarrow[\cong]{(\Phi_{TY}^H)^{-1}} & H^{ev}(Y) \end{array}$$

where Φ_{TX}^K and Φ_{TY}^K are the Thom isomorphisms in twisted K -theory for TX and TY , Φ_{TX}^H and Φ_{TY}^H are the cohomology Thom isomorphisms for TX and TY . Then we have

- (1) The push-forward map in twisted K -theory as established in [CW1]

$$f_!^K : K^0(X, o_X) \rightarrow K^0(Y, o_Y)$$

agrees with

$$(\Phi_{TY}^K)^{-1} \circ (df)_!^K \circ \Phi_{TX}^K.$$

- (2) The push-forward map in cohomology theory $f_*^H : H^{ev}(X) \rightarrow H^{ev}(Y)$ is given by

$$f_*^H = (\Phi_{TY}^H)^{-1} \circ (df)_*^H \circ \Phi_{TX}^H.$$

Denote by U_{TX}^H and U_{TY}^H the cohomological Thom classes for TX and TY . Then under the pull-back of the zero section, $0_X^*(U_{TX}^H) = e(TX)$ and $0_Y^*(U_{TY}^H) = e(TY)$ are the Euler classes for TX and TY respectively.

Let the Pontrjagin classes of $\pi_X : TX \rightarrow X$ be symmetric polynomials in x_1^2, \dots, x_m^2 , then

$$\hat{A}(X) = \prod_{k=1}^m \frac{x_k/2}{\sinh(x_k/2)}.$$

The Chern character defect for the left square in (5.1) is given by

$$(5.2) \quad Ch(\Phi_{TX}^K(a)) = \Phi_{TX}^H(Ch_{\tilde{w}_2(X)}(a)\hat{A}^{-1}(X))$$

for any $a \in K^0(X, o_X)$. Here $\hat{A}(X)$ is the A -hat class of TX .

To prove (5.2), note that

$$\begin{aligned}
& (\Phi_{TX}^H)^{-1} Ch(\Phi_{TX}^K(a)) \\
= & (\Phi_{TX}^H)^{-1} (Ch(\pi_X^*(a) \cdot U_{TX})) && \text{Apply Prop. (4.2)} \\
= & (\Phi_{TX}^H)^{-1} (Ch_{\pi_X^* \tilde{w}_2(X)}(\pi_X^*(a)) \cdot Ch_{\pi_X^* \tilde{w}_2(X)}(U_{TX})) \\
= & (\Phi_{TX}^H)^{-1} (\pi_X^*(Ch_{\tilde{w}_2(X)}(a)) \cdot Ch_{\pi_X^* \tilde{w}_2(X)}(U_{TX})) && \text{Note that } (\Phi_{TX}^H)^{-1} = (\pi_X)_*. \\
= & Ch_{w_2(X)}(a) (\Phi_{TX}^H)^{-1} (Ch_{\pi_X^* \tilde{w}_2(X)}(U_{TX})) && \text{By the projection formula.}
\end{aligned}$$

So the Chern character defect for the square

$$\begin{array}{ccc}
K^0(X, o_X) & \xrightarrow[\cong]{\Phi_{TX}^K} & K_c^0(TX) \\
Ch_{\tilde{w}_2(X)} \downarrow & & \downarrow Ch \\
H^{ev}(X) & \xrightarrow[\cong]{\Phi_{TX}^H} & H_c^{ev}(TX)
\end{array}$$

is given by

$$\mathcal{D}(X) = (\Phi_{TX}^H)^{-1} (Ch_{\pi_X^* \tilde{w}_2(X)}(U_{TX})) \in H^{ev}(X).$$

From the cohomology Thom isomorphism, we have

$$0_X^* \circ \Phi_{TX}^H(\mathcal{D}(X)) = \mathcal{D}(X) e(TX),$$

under the pull-back of the zero section 0_X of the tangent bundle TX .

Therefore, we have

$$\mathcal{D}(X) = \frac{0_X^*(Ch_{\pi_X^* \tilde{w}_2(X)}(U_{TX}))}{e(TX)}.$$

By the construction of the Thom class U_{TX} , under the pull-back of the zero section 0_X of TX , $0_X^*(U_{TX})$ is a twisted K -class in $K^0(X, o_X)$ and

$$\begin{aligned}
& 0_X^* Ch_{\pi_X^* \tilde{w}_2(X)}(U_{TX}) \\
= & Ch_{\tilde{w}_2(X)}(0_X^*(U_{TX})) \\
= & \prod_{k=1}^m (e^{x_k/2} - e^{-x_k/2}).
\end{aligned}$$

Thus, (5.2) follows from

$$\mathcal{D}(X) = \prod_{k=1}^m \frac{(e^{x_k/2} - e^{-x_k/2})}{x_k} = \hat{A}^{-1}(X).$$

This implies that the following diagram commutes

$$\begin{array}{ccc}
K^0(X, o_X) & \xrightarrow[\cong]{\Phi_{TX}^K} & K_c^0(TX) \\
Ch_{\tilde{w}_2(X)}(-) \cdot \hat{A}(X) \downarrow & & \downarrow Ch(-) \cdot \pi_X^* \hat{A}^2(X) \\
H^{ev}(X) & \xrightarrow[\cong]{\Phi_{TX}^H} & H_c^{ev}(TX).
\end{array} \tag{5.3}$$

Similarly, the Chern character defect in

$$\begin{array}{ccc} K^0(Y, o_Y) & \xrightarrow[\cong]{\Phi_{TY}^K} & K_c^0(TY) \\ \downarrow Ch_{\tilde{w}_2(Y)} & & \downarrow Ch \\ H^{ev}(Y) & \xrightarrow[\cong]{\Phi_{TY}^H} & H_c^{ev}(TY) \end{array}$$

is given by

$$Ch(\Phi_{TY}^K(a)) = \Phi_{TY}^H(Ch_{\tilde{w}_2(Y)}(a)\hat{A}^{-1}(Y))$$

for any $a \in K^0(Y, o_Y)$. This implies that the Chern character defect for the right square in (5.1) is given by

$$(5.4) \quad Ch_{\tilde{w}_2(Y)}((\Phi_{TY}^K)^{-1}(c)) \cdot \hat{A}^{-1}(Y) = (\Phi_{TY}^H)^{-1}(Ch(c))$$

for any $c \in K_c^0(TY)$. Hence, we have the following commutative diagram

$$(5.5) \quad \begin{array}{ccc} K_c^0(TY) & \xrightarrow[\cong]{(\Phi_{TY}^K)^{-1}} & K^0(Y, o_Y) \\ \downarrow Ch(-) \cdot \pi_Y^* \hat{A}^2(Y) & & \downarrow Ch_{\tilde{w}_2(Y)}(-) \cdot \hat{A}(Y) \\ H_c^{ev}(TY) & \xrightarrow[\cong]{(\Phi_{TY}^H)^{-1}} & H^{ev}(Y). \end{array}$$

The Chern character for the middle square in (5.1) follows from the Riemann-Roch theorem in ordinary K -theory for the K -oriented map $df : TX \rightarrow TY$ with the orientation given by canonical $Spin^c$ manifolds TX and TY . Note that the Todd classes of $Spin^c$ manifolds of TX and TY are given by $\pi_X^*(\hat{A}^2(X))$ and $\pi_Y^*(\hat{A}^2(Y))$ respectively. This is due to two facts, that $T(TX) \cong \pi_X^*(TX \otimes \mathbb{C})$ and that $Td(TX \otimes \mathbb{C}) = \hat{A}^2(X)$. So we have

$$(5.6) \quad Ch((df)_!^K(a)) \cdot \pi_Y^*(\hat{A}^2(Y)) = (df)_*^H(Ch(a)\pi_X^*(\hat{A}^2(X)))$$

for any $a \in K_c^0(TX)$. Hence, the following diagram commutes

$$(5.7) \quad \begin{array}{ccc} K_c^0(TX) & \xrightarrow{(df)_!^K} & K_c^0(TY) \\ \downarrow Ch(-) \cdot \pi_X^* \hat{A}^2(X) & & \downarrow Ch(-) \cdot \pi_Y^* \hat{A}^2(Y) \\ H_c^{ev}(TX) & \xrightarrow{(df)_*^H} & H_c^{ev}(TY). \end{array}$$

Putting (5.3), (5.5) and (5.7) together, we get the following commutative diagram

$$\begin{array}{ccc} K^0(X, o_X) & \xrightarrow{f_!^K} & K^0(Y, o_Y) \\ \downarrow Ch_{\tilde{w}_2(X)}(-) \cdot \hat{A}(X) & & \downarrow Ch_{\tilde{w}_2(Y)}(-) \cdot \hat{A}(Y) \\ H^{ev}(X) & \xrightarrow{f_*^H} & H^{ev}(Y) \end{array}$$

which leads to

$$Ch_{\tilde{w}_2(Y)}(f_!^K(a))\hat{A}(Y) = f_*^H(Ch_{\tilde{w}_2(X)}(a)\hat{A}(X))$$

for any $a \in K^0(X, o_X)$. This completes the proof of the Riemann-Roch theorem in twisted K -theory. \square

With some notational changes, the above argument can be applied to establish the general Riemann-Roch theorem in twisted K -theory. Let $f : X \rightarrow Y$ a smooth map between oriented manifolds with $\dim Y - \dim X = 0 \pmod 2$. Let $\tilde{\alpha} = (\mathcal{G}_\alpha, \theta, \omega)$ be a differential twisting which lifts $\alpha : Y \rightarrow K(\mathbb{Z}, 3)$, $f^*(\tilde{\alpha})$ is the pull-back differential twisting which lifts $\alpha \circ f : X \rightarrow K(\mathbb{Z}, 3)$. Then we have the following Riemann-Roch formula

$$(5.8) \quad Ch_{\tilde{\alpha}}(f_!^K(a))\hat{A}(Y) = f_*^H(Ch_{f^*\tilde{\alpha} + \tilde{w}_2(Y) + f^*\tilde{w}_2(Y)}(a)\hat{A}(X))$$

for any $a \in K^0(X, \alpha \circ f + o_X + f^*(o_Y))$. In particular, we have the following Riemann-Roch formula for a trivial twisting $\alpha : Y \rightarrow K(\mathbb{Z}, 3)$

$$(5.9) \quad Ch(f_!^K(a))\hat{A}(Y) = f_*^H(Ch_{\tilde{w}_2(Y) + f^*\tilde{w}_2(Y)}(a)\hat{A}(X))$$

for any $a \in K^0(X, o_X + f^*(o_Y))$.

When f is K -oriented, and equipped with a $Spin^c$ structure whose determinant bundle is L , there is a canonical isomorphism

$$\Psi : K^0(X) \cong K^0(X, o_X + f^*(o_Y))$$

such that $Ch_{\tilde{w}_2(Y) + f^*\tilde{w}_2(Y)}(\Psi(a)) = e^{c_1(L)/2}Ch(a)$ for any $a \in K^0(X)$. Then the Riemann-Roch formula (5.9) agrees with the Riemann-Roch formula for K -oriented maps as established in [AH].

6. The twisted index formula

In this Section, we establish the index pairing for a closed smooth manifold with a twisting α

$$K^{ev/odd}(X, \alpha) \times K_{ev/odd}^{\text{an}}(X, \alpha) \longrightarrow \mathbb{Z}$$

in terms of the local index formula for twisted geometric cycles.

THEOREM 6.1. *Let X be a smooth closed manifold with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$. The index pairing*

$$K^0(X, \alpha) \times K_0(X, \alpha) \longrightarrow \mathbb{Z}$$

is given by

$$\langle \xi, (M, \iota, \nu, \eta, [E]) \rangle = \int_M Ch_{\tilde{w}_2(M)}(\eta_*(\iota^*\xi \otimes E))\hat{A}(M)$$

where $\xi \in K^0(X, \alpha)$, and the geometric cycle $(M, \iota, \nu, \eta, [E])$ defines a twisted K -homology class on (X, α) . Here

$$\eta_* : K^*(M, \iota^*\alpha) \cong K^*(M, o_M)$$

is an isomorphism, and $Ch_{\tilde{w}_2(M)}$ is the Chern character on $K^0(M, o_M)$.

PROOF. Recall that the index pairing $K^0(X, \alpha) \times K_0(X, \alpha) \longrightarrow \mathbb{Z}$ can be defined by the internal Kasparov product (Cf. [Kas3] and [ConSka])

$$KK(\mathbb{C}, C(X, \mathcal{P}_\alpha(\mathcal{K}))) \times KK(C(X, \mathcal{P}_\alpha(\mathcal{K})), \mathbb{C}) \longrightarrow KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z},$$

and is functorial in the sense that if $f : Y \rightarrow X$ is a continuous map and Y is equipped with a twisting $\alpha : X \rightarrow \mathbb{Z}$ then

$$\langle f^*b, a \rangle = \langle b, f_*(a) \rangle$$

for any $a \in K_0(Y, f^*\alpha)$ and $b \in K^0(X, \alpha)$.

Note that under the assembly map, the geometric cycle $(M, \iota, \nu, \eta, [E])$ is mapped to $\iota_* \circ \eta_*([M] \cap [E])$, for $\xi \in K^0(X, \alpha)$. Hence, we have

$$\begin{aligned} & \langle \xi, (M, \iota, \nu, \eta, [E]) \rangle \\ &= \langle \xi, \iota_* \circ \eta_*([M] \cap [E]) \rangle \\ &= \langle \iota^* \xi, \eta_*([M] \cap [E]) \rangle \\ &= \langle \eta_*(\iota^* \xi \otimes E), [M] \rangle. \end{aligned}$$

Here $\eta_*(\iota^* \xi \otimes E) \in K^0(M, o_M)$ and $[M]$ is the fundamental class in $K_{ev}^{\text{an}}(M, o_M)$ which is Poincaré dual to the unit element $\underline{\mathbb{C}}$ in $K^0(M)$. The index pairing between $K^0(M, o_M) \times K_{ev}^{\text{an}}(M, o_M)$ can be written as

$$K^0(M, o_M) \times K_{ev}^{\text{an}}(M, o_M) \rightarrow K^0(M, o_M) \times K^0(M) \rightarrow K^0(M, o_M) \rightarrow \mathbb{Z}$$

where the first map is given by the Poincaré duality $K_{ev}^{\text{an}}(M, o_M) \cong K^0(M)$, the middle map is the action of $K^0(M)$ on $K^0(M, o_M)$, and the last map is the push-forward map of $\epsilon : M \rightarrow pt$. Therefore, we have

$$\begin{aligned} & \langle \eta_*(\iota^* \xi \otimes E), [M] \rangle \\ &= \epsilon_!^K(\eta_*(\iota^* \xi \otimes E) \otimes \underline{\mathbb{C}}) \\ &= \epsilon_!^K(\eta_*(\iota^* \xi \otimes E)). \end{aligned}$$

By twisted Riemann-Roch (Theorem 5.3),

$$\begin{aligned} & \epsilon_!(\eta_*(\iota^* \xi \otimes E)) \\ &= \epsilon_*^H(Ch_{\tilde{w}_2(M)}(\eta_*(\iota^* \xi \otimes E)) \hat{A}(M)) \\ &= \int_M Ch_{\tilde{w}_2(M)}(\eta_*(\iota^* \xi \otimes E)) \hat{A}(M). \end{aligned}$$

This completes the proof of the twisted index formula. \square

Note that $\epsilon : M \rightarrow pt$ can be written as $\iota \circ \epsilon_X : M \rightarrow X \rightarrow pt$. Applying the Riemann-Roch Theorem 5.3, we can write the above index pairing as

$$\begin{aligned} & \langle (M, \iota, \nu, \eta, [E]), \xi \rangle \\ &= \int_M Ch_{\tilde{w}_2(M)}(\eta_*(\iota^* \xi \otimes E)) \hat{A}(M) \\ &= \int_X Ch_{\tilde{w}_2(X)}(\iota_!(E) \otimes \xi) \hat{A}(X) \end{aligned}$$

where $\iota_! : K^0(M) \rightarrow K^0(X, -\alpha + o_X)$ is the push-forward map in twisted K -theory,

$$K^0(X, \alpha) \times K^0(X, -\alpha + o_X) \longrightarrow K^0(X, o_X)$$

is the multiplication map (2.7), and

$$Ch_{\tilde{w}_2(X)} : K^0(X, o_X) \longrightarrow H^{ev}(X)$$

is the twisted Chern character (which agrees with the relative Chern character under the identification $K^0(X, o_X) \cong K^0(X, W_3(X))$, the K -theory of Clifford modules on X).

7. Mathematical definition of D-branes and D-brane charges

Here we give a mathematical interpretation of D-branes in Type II string theory using the twisted geometric cycles and use the index theorem in the previous Section to compute charges of D-branes. In Type II superstring theory on a manifold X , a string worldsheet is an oriented Riemann surface Σ , mapped into X with $\partial\Sigma$ mapped to an oriented submanifold M (called a D-brane world-volume, a source of the Ramond-Ramond flux). The theory also has a Neveu-Schwarz B -field classified by a characteristic class $[\alpha] \in H^3(X, \mathbb{Z})$.

In physics, the D-brane world volume M carries a gauge field on a complex vector bundle (called the Chan-Paton bundle), so a D-brane is given by a submanifold M of X with a complex bundle E and a connection ∇^E . This data actually defines a differential K -class

$$[(E, \nabla^E)]$$

in differential K -theory $\check{K}(M)$.

When the B -field is topologically trivial, that is $[\alpha] = 0$, D-brane charge takes values in ordinary K -theory $K^0(X)$ or $K^1(X)$ for Type IIB or Type IIA string theory (as explained in [MM][Wit]). For a D-brane M to define a class in the K -theory of X , its normal bundle ν_M must be endowed with a $Spin^c$ structure. Equivalently, the embedding

$$\iota : M \longrightarrow X$$

is K -oriented so that the push-forward map in K -theory ([AH])

$$\iota_*^K : K^0(M) \longrightarrow K^{ev/odd}(X)$$

is well-defined, (it takes values in even or odd K -groups depending on the dimension of M). So the D-brane charge of $(\iota : M \rightarrow X, E)$ is

$$\iota_*^K([E]) \in K^{ev/odd}(X).$$

It was proposed in [MM] that the cohomological Ramond-Ramond charge of the D-brane is given by

$$\mathcal{Q}_{RR}(\iota : M \rightarrow X, E) = ch(f_*^K(E))\sqrt{\hat{A}(X)}$$

when X is a $Spin$ manifold. A natural K -theoretic interpretation follows from the fact that the modified Chern character isomorphism

$$K_{\mathbb{Q}}^{ev/odd}(X) \longrightarrow H^{ev/odd}(X, \mathbb{Q})$$

given by mapping $a \mapsto ch(a)\sqrt{\hat{A}(X)}$ is an isometry with the natural bilinear pairings on $K_{\mathbb{Q}}^*(X) = K^*(X) \otimes \mathbb{Q}$ and $H^{ev/odd}(X, \mathbb{Q})$. Here the pairing on $K(X)$ is given by the index of the Dirac operator

$$(a, b)_K = \text{Index}(\mathcal{D}_{a \otimes b}) = \int_X ch(a)ch(b)\hat{A}(X) = (ch(a)\sqrt{\hat{A}(X)}, (ch(b)\sqrt{\hat{A}(X)})_H.$$

When the B -field is not topologically trivial, that is $[\alpha] \neq 0$, then $[\alpha]$ defines a complex line bundle over the loop space LX , or a stable isomorphism class of bundle gerbes over X . Then in order to have a well-defined worldsheet path integral, Freed and Witten in [FreWit] showed that

$$(7.1) \quad \iota^*[\alpha] + W_3(\nu_M) = 0.$$

When $\iota^*[\alpha] \neq 0$, that means ι is not K-oriented, then the push-forward map in K -theory ([AH])

$$\iota_!^K : K^0(M) \longrightarrow K^*(X)$$

is **not** well-defined. Witten explained in [Wit] that D-brane charges should take values in a twisted form of K -theory, as supported further by evidence in [BouMat] and [Kap].

In [Wa], the mathematical meaning of (7.1) was discovered using the notion of α -twisted $Spin^c$ manifolds for a continuous map

$$\alpha : X \longrightarrow K(\mathbb{Z}, 3)$$

representing $[\alpha] \in H^3(X, \mathbb{Z})$. When X is $Spin^c$, the datum to describe a D-brane is exactly a geometric cycle for the twisted K -homology $K_{ev/odd}^{\text{geo}}(X, \alpha)$. By Poincaré duality, we have

$$K_{ev/odd}^{\text{geo}}(X, \alpha) \cong K^0(X, \alpha + o_X)$$

with the orientation twisting $o_X : X \rightarrow K(\mathbb{Z}, 3)$ trivialized by a choice of a $Spin^c$ structure. Hence,

$$K^0(X, \alpha + o_X) \cong K^0(X, \alpha).$$

For a general manifold X , a submanifold $\iota : M \rightarrow X$ with

$$\iota^*([\alpha]) + W_3(\nu_M) = 0,$$

then there is a homotopy commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\nu_M} & \mathbf{BSO} \\ \downarrow \iota & \eta \swarrow \text{=} & \downarrow W_3 \\ X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3), \end{array}$$

here ν_M also denotes a classifying map of the normal bundle, or a classifying map of the bundle $TM \oplus \iota^*TX$. This motivates the following definition (see also [CW2]).

DEFINITION 7.1. Given a smooth manifold X with a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$, a B -field of (X, α) is a differential twisting lifting α

$$\check{\alpha} = (\mathcal{G}_\alpha, \theta, \omega),$$

which is a (lifting, or local) bundle gerbe \mathcal{G}_α with a connection θ and a curving ω . The field strength of the B -field $(\mathcal{G}_\alpha, \theta, \omega)$ is given by the curvature H of $\check{\alpha}$.

A Type II (generalized) D-brane in (X, α) is a complex vector bundle E with a connection ∇^E over a twisted $Spin^c$ manifold M . The twisted $Spin^c$ structure on M is given by the following homotopy commutative diagram together with a choice of a homotopy η

$$(7.2) \quad \begin{array}{ccc} M & \xrightarrow{\nu_\iota} & \mathbf{BSO} \\ \downarrow \iota & \eta \swarrow \text{=} & \downarrow W_3 \\ X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3) \end{array}$$

where ν_ι is the classifying map of $TM \oplus \iota^*TX$.

REMARK 7.2. The twisted $Spin^c$ manifold M in Definition 7.1 is the D-brane world volume in Type II string theory. The twisted $Spin^c$ structure given in (7.2) implies that D-brane world volume $M \subset X$ in Type II string theory satisfies the Freed-Witten anomaly cancellation condition

$$\iota^*[\alpha] + W_3(\nu_M) = 0.$$

In particular, if the B -field of (X, α) is topologically trivial, then the normal bundle of $M \subset X$ is equipped with a $Spin^c$ structure given by (7.2).

Given a Type II D-brane $(M, \iota, \nu_\iota, \eta, E, \nabla^E)$, the homotopy η induces an isomorphism

$$\eta_* : K^0(M) \rightarrow K^0(M, \iota^* \alpha + o_\iota).$$

Here o_ι denotes the orientation twisting of the bundle $TM \oplus \iota^* TX$. Note that

$$\iota_!^K : K^0(M, \iota^* \alpha + o_\iota) \longrightarrow K^{en/odd}(X, \alpha)$$

is the pushforward map (2.9) in twisted K -theory. Hence we have a canonical element in $K^{en/odd}(X, \alpha)$ defined by

$$\iota_!^K(\eta_*([E])),$$

called the D-brane charge of $(M, \iota, \nu_\iota, \eta, E)$. We remark that a Type II D-brane

$$(M, \iota, \nu_\iota, \eta, E, \nabla^E)$$

defines an element in differential twisted K -theory $\check{K}^{en/odd}(X, \check{\alpha})$.

From (7.2), we know that M is an $(\alpha + o_X)$ -twisted $Spin^c$ manifold as we have the following homotopy commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\nu} & \mathbf{BSO} \\ \downarrow \iota & \swarrow \eta & \downarrow w_3 \\ X & \xrightarrow{\alpha + o_X} & K(\mathbb{Z}, 3) \end{array}$$

where ν is the classifying map of the stable normal bundle of M . Together with the following proposition, we conclude that the Type II D-brane charges, in the present of a B -field

$$\check{\alpha} = (\mathcal{G}_\alpha, \theta, \omega),$$

are classified by twisted K -theory $K^0(X, \alpha)$.

PROPOSITION 7.3. *Given a twisting $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ on a smooth manifold X , every twisted K -class in $K^{ev/odd}(X, \alpha)$ is represented by a geometric cycle supported on an $(\alpha + o_X)$ -twisted closed $Spin^c$ -manifold M and an ordinary K -class $[E] \in K^0(M)$.*

For completeness, we also give a definition of Type I D-branes (Cf. [MMS], [RSV] and Section 8 in [Wa]).

DEFINITION 7.4. Given a smooth manifold X with a KO -twisting $\alpha : X \rightarrow K(\mathbb{Z}_2, 2)$, a Type I (generalized) D-brane in (X, α) is a real vector bundle E with a connection ∇^E over a twisted $Spin$ manifold M . The twisted $Spin$ structure on M is given by the following homotopy commutative diagram together with a choice of a homotopy η

$$(7.3) \quad \begin{array}{ccc} M & \xrightarrow{\nu_\iota} & \mathbf{BSO} \\ \downarrow \iota & \swarrow \eta & \downarrow w_2 \\ X & \xrightarrow{\alpha} & K(\mathbb{Z}_2, 2) \end{array}$$

where w_2 is the classifying map of the principal $K(\mathbb{Z}_2, 1)$ -bundle $\mathbf{BSpin} \rightarrow \mathbf{BSO}$ associated to the second Stiefel-Whitney class, η is a homotopy between $w_2 \circ \nu_\iota$ and $\alpha \circ \iota$. Here ν_ι is the classifying map of $TM \oplus \iota^* TX$.

REMARK 7.5. A Type I D-brane in (X, α) has its support on a manifold M if and only if there is a differentiable map $\iota : M \rightarrow X$ such that

$$\iota^*([\alpha]) + w_2(\nu_\iota) = 0.$$

Here ν_ι denotes the bundle $TM \oplus \iota^*TX$.

Given a Type I D-brane in (X, α) , the push-forward map in twisted KO-theory

$$KO^*(M) \xrightarrow[\cong]{\eta^*} KO^*(M, \alpha \circ \iota + o_\iota) \xrightarrow{\iota_!^{KO}} KO^*(X, \alpha)$$

defines a canonical element in $KO^*(X, \alpha)$. Every class in $KO^*(X, \alpha)$ can be realized by a Type I (generalized) D-brane in (X, α) . Hence, we conclude that the Type I D-brane charges are classified by twisted KO-theory $KO^{ev/odd}(X, \alpha)$.

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